The Integrated Density of States for Random Schrödinger Operators

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Dedicated to Barry Simon on the occasion of his 60th birthday

Abstract. We survey some aspects of the theory of the integrated density of states (IDS) of random Schrödinger operators. The first part motivates the problem and introduces the relevant models as well as quantities of interest. The proof of the existence of this interesting quantity, the IDS, is discussed in the second section. One central topic of this survey is the asymptotic behavior of the integrated density of states at the boundary of the spectrum. In particular, we are interested in Lifshitz tails and the occurrence of a classical and a quantum regime. In the last section we discuss regularity properties of the IDS. Our emphasis is on the discussion of fundamental problems and central ideas to handle them. Finally, we discuss further developments and problems of current research.

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1. Introduction

1.1. Physical Background: Models and Notation. The time evolution of a quantum mechanical state ψ is obtained from the time dependent Schrödinger equation

$$i\,\frac{\partial}{\partial t}\,\psi \;=\; H\,\psi. \tag{1.1}$$

Another important differential equation is the heat or diffusion equation

$$\frac{\partial}{\partial t}\,\psi \;=-\,H\,\psi. \tag{1.2}$$

In order to investigate either of these equations, it is extremely useful to know as much as possible about the operator H and its spectrum. In general the Schrödinger operator H is of the form

$$H = H_0 + V.$$

The free operator H_0 represents the kinetic energy of the particle. In the absence of magnetic fields, it is given by the Laplacian

$$H_0 = -\,\Delta = -\sum_{\nu=1}^{d} \frac{\partial^2}{\partial {x_\nu}^2}.$$

The potential V encoding the forces $F(x) = -\nabla V(x)$ is acting as a multiplication operator in the Hilbert space $L^2(\mathbb{R}^d)$.

Occasionally, we will replace H_0 with the operator $H_0(B)$ which contains a homogeneous magnetic field. For $d = 2$ $H_0(B)$, $B > 0$, is given by

$$H_0(B) = \left(i\frac{\partial}{\partial x_1} - \frac{1}{2}Bx_2\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{1}{2}Bx_1\right)^2.$$

In contrast to $H_0(B = 0)$, the spectrum of $H_0(B)$ for $B > 0$ is a countable set $\sigma(H_0(B)) = \{(2n + 1)B; n \in \mathbb{N}\}$. The energies $E_n = (2n + 1)$ are eigenvalues of $H_0(B)$ of infinite multiplicity, called the Landau levels.

1.2. Random Potentials. First-order physical modeling often assumes an ideal background, e.g., a homogeneous material without any impurities. For example, in ideal crystals the atoms or nuclei are supposed to be distributed on a periodic lattice (say the lattice $\mathbb{Z}^d$) in a completely regular way. We assume that a particle (electron) at the point $x \in \mathbb{R}^d$ feels a potential of the form $q\,f(x - i)$ due to an atom (or ion or nucleus) located at the point $i \in \mathbb{Z}^d$. We call the function f the *single site potential*. The coupling constant q represents the charge of the particle at the lattice point i. So, in a regular crystal, the particle is exposed to a total potential

$$V(x) = \sum_{i\in\mathbb{Z}^d} q\,f(x - i). \tag{1.3}$$

The potential V in (1.3) is periodic with respect to the lattice $\mathbb{Z}^d$, i.e., $V(x-i) = V(x)$ for all $x \in \mathbb{R}^d$ and $i \in \mathbb{Z}^d$. The mathematical theory of Schrödinger operators with periodic potentials is well developed (see, e.g., [**35, 112, 150**]). It is based on a thorough analysis of the symmetry properties of periodic operators. The spectrum of periodic Schrödinger operators has a band structure and the spectrum is purely absolutely continuous, i.e.,

$$\sigma(H) = \bigcup_{n=0}^{\infty} [a_n, b_n] \qquad \text{with} \quad a_n < b_n \leq a_{n+1}, \tag{1.4}$$

$$\sigma_{\mathrm{sing}}(H) = \emptyset, \qquad \sigma(H) = \sigma_{\mathrm{ac}}(H). \tag{1.5}$$

The real world is not ideal. Solids occur in nature in various forms. Sometimes they are (almost) totally ordered, sometimes they are more or less completely disordered. For a mathematical modelling of disordered solids two ingredients are essential: the spatial homogeneity in the mean and the disappearance of long range correlations. In full generality these properties are studied in the theory of ergodic operators. This class of operators contains, e.g., quasicrystals and can also be related to random matrices. In this publication we will be concerned with *random* Schrödinger operators.

The most popular and best understood model of a disordered solid is the alloy-type Schrödinger operator. It models a mixture of different atoms located at lattice positions. The type of atom at the lattice point i is assumed to be random. These particles are represented by randomly distributed coupling constants q_i encoding the different charges. The total potential is given by

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} q_i(\omega)\, f(x-i). \tag{1.6}$$

When talking about the alloy-type model, we will mean (1.6) with the following assumptions:

(1) The single site potential f is bounded, non-negative and strictly positive on an open set.
(2) f satisfies $f(x) \leq C\,(1+|x|)^{-(d+\varepsilon)}$ for some C and $\varepsilon > 0$.
(3) The random variables q_i are independent and identically distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
(4) The common probability distribution of the q_i is denoted by P_0. Its support $\operatorname{supp} P_0$ is compact and contains at least two points.

Assumption (1) can be considerably relaxed. For example, for most of the following results one may allow local singularities for f. By assuming (1) we avoid technical difficulties which may obscure the main argument. More details on weaker conditions can be found in the papers cited. Assumption (2) ensures that the sum in (1.6) is convergent. The compactness of $\operatorname{supp} P_0$ is convenient but not always necessary and in some especially marked situations we consider also unbounded single site distributions. However, for many of our results we need that the q_i (and hence $\operatorname{supp} P_0$) are bounded

below. The physical model suggests that $\operatorname{supp} P_0$ consists of finitely many points only (the possible charges). However, for many mathematical results it is necessary (or at least convenient) to suppose that P_0 has a (bounded) density g, i.e., $P_0 = g(\lambda)\, d\lambda$.

One might argue that such an assumption is acceptable as a purely technical one. On the other hand one could argue the problem is not understood as long as it is impossible to handle the physically relevant case of finitely many values.

A simplified version of the alloy-type Schrödinger operator above is the (discrete) Anderson model. Here the Hilbert space is the sequence space $\ell^2(\mathbb{Z}^d)$ instead of $L^2(\mathbb{R}^d)$ and the free operator H_0 is replaced by the discrete analogue of the Laplacian. This is the finite-difference operator

$$(h_0\, u)(n) = - \sum_{|m-n|=1} (u(m) - u(n)). \tag{1.7}$$

Above we set $|n| = \sum_{i=1}^{d} |n_i|$ on $\mathbb{Z}^d$. The potential is a multiplication operator $V = V_\omega$ on $\ell^2(\mathbb{Z}^d)$ with $V_\omega(i)$ independent, identically distributed, and the total Hamiltonian is defined by $h = h_0 + V_\omega$. We will call this setting the "discrete case" in contrast to Schrödinger operators on $L^2(\mathbb{R}^d)$ which we refer to as the "continuous case."

The most frequently used and most important approach to model *amorphous material* like glass or rubber defines the random locations of the atoms by a Poisson random measure. This random point measure can be characterized by the number $n_A = \mu_\omega(A)$ of random points in the set A. We assume that the random variables n_A and n_B are independent for disjoint (measurable) sets A, B and $\mathbb{P}\left(n_A = k\right) = \frac{|A|^k}{k!}\, e^{-|A|}$ ($|A|$ denotes the Lebesgue measure of A). With this notation we may write the Poisson potential for an amorphous solid as

$$V_\omega(x) = \int_{\mathbb{R}^d} q\, f(x-\eta)\, d\,\mu_\omega(\eta). \tag{1.8}$$

To model thin disordered layers, we also consider random potentials which are concentrated along a hypersurface in $\mathbb{R}^d$ (or $\mathbb{Z}^d$). For example, we are going to consider "surface" alloy potentials. To define such a potential let us write $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ then

$$V_\omega(x_1, x_2) = \sum_{i_1 \in \mathbb{Z}^{d_1}} q_{i_1}(\omega) f(x_1 - i_1, x_2)$$

is a random potential which is concentrated along the hypersurface $\mathbb{R}^{d_1}$ in $\mathbb{R}^d$. In addition, there may be a random or periodic background potential on $\mathbb{R}^d$.

Most of the theorems we are going to discuss can be proved for rather general single site potentials f and probability distributions P_0 of the q_i. For example, most of the time we can allow some local singularities of f. To

simplify the following discussion, we will assume in this paper the conditions defined in the context of the alloy-type potential.

The above random operators are examples of "ergodic operators." This class of operators includes not only most random operators but also periodic and almost periodic operators. Most of the results of Section 2 and part of Section 4 can be shown for general ergodic operators. We refer to [**148, 15, 27, 119, 166**] and [**68**] for a discussion of this general context.

1.3. The Concept of the Integrated Density of States. The (integrated) density of states is a concept of fundamental importance in condensed matter physics. It measures the "number of energy levels per unit volume" near (resp. below) a given energy.

Typical systems arising in solid state physics have periodic or ergodic potentials. Consequently, the spectrum of the corresponding Hamiltonian is not discrete. Therefore, we cannot just count the eigenvalues below E or within an interval $[E_1, E_2]$. On the other hand, the number of electrons in such a system, which extends to infinity, ought to be infinite. For these two reasons, the Pauli exclusion principle does not make immediate sense. (How do we distribute infinitely many electrons on a continuum of spectral energies?)

However, there may be a chance to make sense out of the Pauli principle by first restricting the system to a finite volume Λ. Inside Λ there should be only finitely many electrons, in fact, we may assume that the number of electrons in a given Λ is proportional to the volume $|\Lambda|$ of this set.

If P is a finite-dimensional orthogonal projection, then $\mathrm{tr}(P)$ is the dimension of its range. If $P_{(-\infty,E]}$ is the spectral projection of a random Schrödinger operator (which as a rule has infinite-dimensional range) and if Λ_L is a cube of side length L around the origin, then we may call $\mathrm{tr}(\chi_{\Lambda_L} P_{(-\infty,E]})$ the restriction of $P_{(-\infty,E]}$ to the cube Λ_L. Above χ_A is the characteristic function of the set A. Thus, we may try to define the integrated density of states as

$$N(E) \;=\; \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\ \mathrm{tr}\,\big(\chi_{\Lambda_L} P_{(-\infty,E]}\big). \tag{1.9}$$

Of course, we have to prove that the limit in (1.9) does exist and is not trivial. We will deal with these questions in Section 2.

There is another way to define the integrated density of states which turns out to be equivalent to (1.9): We restrict the operator H_ω to the Hilbert space $L^2(\Lambda)$. To obtain a self adjoint operator we have to impose boundary conditions at $\partial\Lambda$, e.g., Dirichlet or Neumann boundary conditions. We call the corresponding operators H_Λ^D and H_Λ^N, respectively. These operators have compact resolvents, i.e., their spectra are purely discrete. We denote by

$$E_1(H_\Lambda^D) \le E_2(H_\Lambda^D) \le E_3(H_\Lambda^D) \dots \tag{1.10}$$

the eigenvalues of H_Λ^D (and analogously for H_Λ^N) in increasing order, where eigenvalues are repeated according to their multiplicity. The eigenvalue counting function of an operator A with purely discrete spectrum is defined by

$$N(A,E) = \#\{\, n \,|\, E_n(A) \leq E\} \;=\; \mathrm{tr}\,\big(P_{(-\infty,E]}(A)\big). \tag{1.11}$$

Analogously to (1.9), we can therefore define

$$\begin{aligned} N^D(E) &= \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\, N(H_{\Lambda_L}^D, E) \\ &= \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\; \mathrm{tr}\,\big(P_{(-\infty,E]}(H_{\Lambda_L}^D)\big) \end{aligned} \tag{1.12}$$

and similarly for Neumann boundary conditions,

$$\begin{aligned} N^N(E) &= \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\, N(H_{\Lambda_L}^N, E) \\ &= \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\; \mathrm{tr}\,\big(P_{(-\infty,E]}(H_{\Lambda_L}^N)\big). \end{aligned} \tag{1.13}$$

This procedure to define the integrated density of states makes sense only if N, N^D and N^N all *exist* and *agree*.

This is, indeed, the case. We will see in the sequel that each of these definitions has its own technical advantage. The integrated density of states N is basic for studying the physical (in particular the thermodynamical) properties of disordered systems. ¿¿From a mathematical point of view, the properties of N are interesting in their own respect. Moreover, properties of N constitute an essential input to prove localization properties of the system.

It is the aim of this review to discuss some of the problems and results connected with the integrated density of states. In Section 2 we sketch the proof of the existence of the integrated density of states and discuss some fundamental questions concerning the probabilistic and the functional analytic approach. In Section 3 we study the behavior of the integrated density of states at the boundary of the spectrum. In the last section we discuss some basic ideas concerning the regularity of the integrated density of states.

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2. The Density of States Measure: Existence

2.1. Introduction and Historical Remarks. The first existence proofs for the integrated density of states go back at least to Pastur. See [**145**] for an early review of the subject.

There are a couple of methods to prove the existence of the integrated density of states. One of them, invented and used by Pastur, is based on the Laplace transform of the integrated density of states and of its approximants. For this method, one proves the convergence of the Laplace transform and uses the fact that convergence of the Laplace transform implies the vague convergence of the underlying measures.

To prove the convergence of the Laplace transforms, it is useful to express the Laplace transform of the finite-volume quantities using the Feynman–Kac-representation of the Schrödinger semigroup e^{-tH}. Feynman–Kac and Laplace transform methods are also used to prove the equivalence of the definitions of the integrated density of states (1.9) and (1.12), (1.13) with either Neumann or Dirichlet (or more general) boundary conditions (see, e.g., Pastur [**145**] or [**74, 34, 68**]).
The definition of the integrated density of states via (1.9) was used by Avron and Simon in the context of almost periodic potentials [**3**]. They also proved that the spectrum of the operator coincides with the growth points of the integrated density of states. In Section 2.2 we will follow this approach to prove the existence of the integrated density of states.

One of the virtues of the definition of the integrated density of states via boundary conditions (1.12) and (1.13) is the fact that they allow lower and upper bounds of N "for free." In fact, one way to examine the behavior of N at the bottom (or top $= \infty$) of the spectrum is based on this approach. We will discuss this approach in Section 2.3 and the estimates based on it in Section 3.

As a rule, quantitative estimates on the effect of introducing boundary conditions are hard to obtain. For example, if one investigates the behavior of N at internal spectral edges it seems extremely difficult to control the perturbation of eigenvalues due to boundary conditions. Klopp [**97**] proposed an approximation of the random potential by periodic ones with growing period. This way we lose monotonicity which is at the heart of the Neumann–Dirichlet approach. Instead one can prove that the approximation is exponentially fast thus gaining good estimates of the remainder.

Finally, we would like to mention that one can also define the integrated density of states via Krein's spectral shift function. This reasoning is well known in scattering theory (see, e.g., [**9, 180**]). In connection with random Schrödinger operators, the spectral shift function was first used by Simon [**159**] to investigate spectral averaging. Kostrykin and Schrader [**106**] applied this technique to prove the existence of the integrated density of states and the density of surface states. This method turns out to be useful also to investigate regularity properties of the integrated density of states [**107**].

The results of this section are true not only for the specific random potentials discussed in Section 1, but rather for general ergodic operators. In fact, the proofs carry over to this general setting in most cases. We refer to the survey [**68**] for details.

2.2. The Existence of the Integrated Density of States. In this section we prove the existence of the integrated density of states as defined in (1.9). To do so, we need little more than Birkhoff's ergodic theorem (see, e.g., [**113**]). Below, as in the rest of this paper, we denote by $\mathbb{E}$ the expectation with respect to the probability measure $\mathbb{P}$.

PROPOSITION 2.1. *If φ is a bounded measurable function of compact support, then*

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \operatorname{tr}(\varphi(H_\omega)\chi_{\Lambda_L}) = \mathbb{E}\left(\operatorname{tr}(\chi_{\Lambda_1}\varphi(H_\omega)\chi_{\Lambda_1})\right) \tag{2.1}$$

for $\mathbb{P}$-almost all ω.

PROOF. Define $\xi_i = \operatorname{tr}(\varphi(H_\omega)\chi_{\Lambda_1(i)})$. ξ_i is an ergodic sequence of random variables. Hence, Birkhoff's ergodic theorem implies that

$$\frac{1}{|\Lambda_L|} \operatorname{tr}(\varphi(H_\omega)\chi_{\Lambda_L}) = \frac{1}{|\Lambda_L|} \sum_{i\in\Lambda_L} \xi_i$$

converges to its expectation value. □

The right-hand side of (2.1) as well as $|\Lambda_L|^{-1} \operatorname{tr}(\varphi(H_\omega)\chi_{\Lambda_L})$ are positive linear functionals on the bounded, continuous functions. They define positive measures ν and ν_L by

$$\int \varphi(\lambda)d\nu(\lambda) = \mathbb{E}(\left(\operatorname{tr}(\chi_{\Lambda_1}\varphi(H_\omega)\chi_{\Lambda_1})\right))$$

and

$$\int_{\mathbb{R}} \varphi(\lambda)\, d\nu_L(\lambda) = \frac{1}{|\Lambda_L|} \operatorname{tr}(\varphi(H_\omega)\chi_{\Lambda_L}).$$

Equation (2.1) suggests that the measures ν_L might converge to the limit measure ν as $L \to \infty$ in the sense of vague convergence of measures. The problem is (2.1) holds only for fixed φ on a set Ω_φ of full probability; respectively (2.1) holds for *all* φ for $\omega \in \bigcap_\varphi \Omega_\varphi$. However, this is an *uncountable* intersection of sets of probability one. The problem is solved by approximating $C_0(\mathbb{R})$ by a countable, dense subset.

THEOREM 2.2. *The measures ν_L converge vaguely to the measure ν $\mathbb{P}$-almost surely, i.e., there is a set Ω_0 of probability one, such that*

$$\int \varphi(\lambda)d\nu_L(\lambda) \to \int \varphi(\lambda)d\nu(\lambda) \tag{2.2}$$

for all $\varphi \in C_0(\mathbb{R})$, the set of continuous functions with compact support, and all $\omega \in \Omega_0$.

DEFINITION 2.3. *The non-random probability measure ν is called the density of states measure. The distribution function N of ν, defined by*

$$N(E) = \nu(]-\infty, E]),$$

is known as the integrated density of states.

Using Theorem 2.2 it is not hard to see:

PROPOSITION 2.4 ([**3**]). $\operatorname{supp}(\nu) = \Sigma\, [= \sigma(H_\omega) \quad a.s.]$.

2.3. Existence via Dirichlet–Neumann-Bracketing. Our first approach to define the density of states measure was based on the additivity of $\operatorname{tr}(\varphi(H_\omega)\chi_{\Lambda_L})$ and the ergodic theorem by Birkhoff. This very naturally fits in the concept of self-averaged quantities from physics.

However, for some part of the further analysis, an alternative approach—the Dirichlet–Neumann bracketing—is more suitable. Let $(H_\omega)_\Lambda^N$ and $(H_\omega)_\Lambda^D$ be the restrictions of H_ω to $L^2(\Lambda)$ with Neumann and Dirichlet boundary conditions. See, e.g., [**150**] for an appropriate definition of these boundary conditions via quadratic forms. Furthermore, we define (for $X = N$ or D, and $E \in \mathbb{R}$)

$$N_\Lambda^X(E) \;:=\; N(\,(H_\omega)_\Lambda^X, E) \;\;=\;\; \operatorname{tr}(\chi_{(-\infty,E]}(H_{\omega\,\Lambda}^{\;X})). \tag{2.3}$$

Our aim is to consider the limits

$$N^X(E) \;\;=\;\; \lim_{L\to\infty} \;\frac{1}{|\Lambda_L|}\, N_{\Lambda_L}^X(E). \tag{2.4}$$

The quantities N_Λ^D and N_Λ^N as defined in (2.3) are distribution functions of point measures ν_Λ^D and ν_Λ^N concentrated in the eigenvalues of H_Λ^D and H_Λ^N, i.e.,

$$N_\Lambda^X(E) \;=\; \nu_\Lambda^X\big((-\infty, E]\big). \tag{2.5}$$

The convergence in (2.4) is meant as the vague convergence of the corresponding measures or, what is the same, as the pointwise convergence of the distribution function $\frac{1}{|\Lambda|}\, N_\Lambda^X$ at all continuity points of the limit.

Let us first look at $\frac{1}{|\Lambda|}\, N_\Lambda^D(E)$. The random field N_Λ^D is *not* additive in Λ, so that we *cannot* use Birkhoff's ergodic theorem. However, N_Λ^D is *superadditive*, in the sense that $N_\Lambda^D\,(E) \geq\; N_{\Lambda_1}^D(E) \;+\; N_{\Lambda_2}^D(E)$ whenever $\Lambda = \Lambda_1 \cup \Lambda_2$ with $(\Lambda_1)^\circ \cap (\Lambda_2)^\circ = \emptyset$. ($M^\circ$ denotes the interior of the set M.) Similarly, N_Λ^N is *subadditive*, i.e., $-N^N$ is superadditive.

THEOREM 2.5. *N_Λ^D is superadditive and N_Λ^N is subadditive. More precisely, if $\Lambda = \Lambda_1 \cup \Lambda_2$ and $(\Lambda_1)^\circ \cap (\Lambda_2)^\circ = \emptyset$ then*

$$N_{\Lambda_1}^D(E) + N_{\Lambda_2}^D(E) \leq N_\Lambda^D(E) \leq N_\Lambda^N(E) \leq N_{\Lambda_1}^N(E) + N_{\Lambda_2}^N(E).$$

Fortunately, there are sub- and superadditive versions of the ergodic theorem, going back at least to Kingman [**66**]. The situation here is ideal for the superadditive ergodic theorem by Akcoglu and Krengel [**2**]. Indeed, one can prove that (for fixed E) the processes $N^D_\Lambda(E)$ and $N^N_\Lambda(E)$ are superadditive and subadditive random fields in the sense of [**2**] respectively (see [**74, 111**]). This yields the following result.

THEOREM 2.6 ([**74**]). *The limits*

$$\bar{N}^D(E) = \lim_{L\to\infty} \quad \frac{1}{|\Lambda_L|}\ N(H^D_{\omega\,\Lambda}, E)$$

and

$$\bar{N}^N(E) = \lim_{L\to\infty} \quad \frac{1}{|\Lambda_L|}\ N(H^N_{\omega\,\Lambda}, E)$$

exist P-almost surely. Moreover,

$$\bar{N}^D(E) = \sup_L \quad \frac{1}{|\Lambda_L|}\ \mathbb{E}\left(N(H^D_{\omega\,\Lambda_L}, E)\right)$$
$$\bar{N}^N(E) = \inf_L \quad \frac{1}{|\Lambda_L|}\ \mathbb{E}\left(N(H^N_{\omega\,\Lambda_L}, E)\right).$$

The functions $\bar{N}^X$ are increasing functions. However, it is not clear whether they are right continuous. To obtain *distribution functions*, we define N^X by making the $\bar{N}^X$ right continuous

$$N^D(E) = \inf_{E'>E}\ \bar{N}^D(E') \tag{2.6}$$
$$N^N(E) = \inf_{E'>E}\ \bar{N}^N(E'). \tag{2.7}$$

Note that $\bar{N}^X$ and N^X disagree at most on a countable set. Since N^D are N^N are distribution functions, they define measures by

$$\nu^D\big((a,b]\big) = N^D(b) - N^D(a) \tag{2.8}$$
$$\nu^N\big((a,b]\big) = N^N(b) - N^N(a). \tag{2.9}$$

¿¿From Theorem 2.6 we obtain the following corollary, which we will use to investigate the asymptotic behavior of the integrated density of states (e.g., for small E).

COROLLARY 2.7. *For any Λ,*

$$\frac{1}{|\Lambda|}\ \mathbb{E}\left(N(H^D_{\omega\Lambda_L}, E\ )\right) \le\ \bar{N}^D(E)\ \le\ \bar{N}^N(E)\ \le\ \frac{1}{|\Lambda|}\ \mathbb{E}\left(N(H^N_{\omega\Lambda_L}, E)\right).$$

Our physical intuition would lead to the hope that $N(E) = N^D(E) = N^N(E)$ since, after all, the introduction of boundary conditions was a mathematical artifact that should not play any role for the final physically meaningful quantity. This is, in fact, true under fairly weak conditions (see [**145, 74, 34**] and references given there).

THEOREM 2.8. *The distribution functions $N(E)$, $N^D(E)$ and $N^N(E)$ agree.*

Theorem 2.8 follows from Theorem 2.10 in the next section. An alternative proof for the Anderson model can be found in the review [**70**]. Theorem 2.8 implies a fortiori that the quantities $\frac{1}{|\Lambda_L|}\ N^D_{\Lambda_L}(E)$ and $\frac{1}{|\Lambda_L|}\ N^D_{\Lambda_L}(E)$ converge to the same limit, except for a countable set of energies E. The exceptional points, if any, are the discontinuity points of N. We will discuss continuity (and, more generally, regularity) properties of N in Section 4.

2.4. A Feynman–Kac Representation for N. In this section we will consider the Laplace transform of the integrated density of states (both $N(E)$ and $N^X(E)$). The Laplace transform of a measure ν with distribution function F is defined by

$$\widetilde{\nu}(t) = \widetilde{F}(t) := \int e^{-\lambda t}\, d\nu(\lambda) \,=\, \int e^{-\lambda t}\, dF(\lambda). \tag{2.10}$$

There is a very useful representation of the Laplace transform of N (and of N^X) via the Feynman–Kac formula. Using this representation one can show that N and N^X are, indeed, the same quantities. Moreover, the Feynman–Kac-representation of N is very useful to compute the asymptotic behavior of N for small or large energies.

The key ingredients of the representation formula for $\widetilde{N}(t)$ are the Brownian motion, the Brownian bridge and the Feynman–Kac-formula. For material about these concepts, we refer to Reed–Simon [**149, 150**] and Simon [**154**].

By $\mathbb{P}^{t,y}_{0,x}$ we denote the measure underlying a Brownian bridge starting in the point $x \in \mathbb{R}^d$ at time 0 and ending at time t in the point y. $\mathbb{E}^{t,y}_{0,x}$ denotes integration over $\mathbb{P}^{t,y}_{0,x}$. A Brownian bridge is a Brownian motion b conditioned on $b(t) = y$. Note that $\mathbb{P}^{t,y}_{0,x}$ is not a probability measure. $\mathbb{P}^{t,y}_{0,x}$ has total mass $p(t,x,y)$ where p denotes the probability kernel of the Brownian motion.

THEOREM 2.9 (Feynman–Kac formula). *If $V \in L^P_{\mathrm{loc,\,unif}}(\mathbb{R}^d)$ for $p = 2$ if $d \le 3$, $p > d/2$ if $d \ge 3$, then e^{-tH} has a jointly continuous integral kernel given by*

$$\begin{aligned} e^{-tH}(x,y) &= \mathbb{E}^{t,y}_{0,x}(e^{-\int_0^t V(b(s))\,ds}) \\ &= \int e^{-\int_0^t V(b(s))\,ds} d\mathbb{P}^{t,y}_{0,x}(b). \end{aligned} \tag{2.11}$$

The integration here is over paths $b(.) \in C([0,t])$.

We remind the reader that we always assume bounded potentials so that the conditions in Theorem 2.9 are satisfied. In the context of the density of states, we are interested in a Feynman–Kac formula for Hamiltonians on

bounded domains. Let us denote by Ω_Λ^t the set of all paths staying inside Λ up to time t, i.e.,

$$\Omega_\Lambda^t = \{b \in C([0,t] : |b(s) \in \Lambda \text{ for all } 0 \le s \le t\}.$$

Then $e^{-tH_\Lambda^D}$ is simply given by restricting the integration in (2.11) to the set Ω_Λ^t, e.g.,

$$e^{-tH_\Lambda^D}(x,y) = \mathbb{E}_{0,x}^{t,y}\big(e^{-\int_0^t V(b(s))\,ds}\chi_{\Omega_\Lambda^t}\big).$$

A proof can be found in Simon [**154**] and Aizenman–Simon [**1**]. There is also a Feynman–Kac formula for Neumann boundary conditions (see [**74**] and references given there).

Now we are able to state the probabilistic representation of the density of states measure in terms of Brownian motion.

THEOREM 2.10. *The Laplace transforms of $N(E)$, $N^D(E)$ and $N^N(E)$ agree and are given by*

$$\widetilde{N}(t) = \widetilde{N}^D(t) = \widetilde{N}^N(t) = \mathbb{E} \times \mathbb{E}_{0,0}^{t,0}(e^{-\int_0^t V_\omega(b(s))ds}). \tag{2.12}$$

For a detailed proof see, e.g., [**68**] and references there. The first step in the proof is to check that the right-hand side of (2.12) is finite for all $t \ge 0$. Interchanging the expectation values with respect to random potential and the Brownian motion, this follows from Jensen's inequality.

The second step of the proof is to compare (2.12) and the Laplace transforms of the approximating density of states measure ν_L and ν_L^D. To prove the theorem one has to estimate the hitting probability of the boundary of Λ for a Brownian motion starting and ending far away from the boundary. Using standard facts of Brownian motion, this tends to 0 in the limit $|\Lambda| \to \infty$.

Once we know that the Laplace transforms of N, N^D and N^N agree, it follows from the uniqueness of the Laplace transform that N, N^D and N^N agree themselves (see, e.g., [**40**]).

2.5. The Density of Surface States. We would like to define a density of states measure for surface potentials as well. Suppose we have a surface potential of the form

$$V_\omega^s(x_1,x_2) = \sum_{i_1 \in \mathbb{Z}^{d_1}} q_{i_1}(\omega) f(x_1 - i_1, x_2)$$

where, as above, $x \in \mathbb{R}^d$ is written as $x = (x_1, x_2)$ with $x_1 \in \mathbb{R}^{d_1}, x_2 \in \mathbb{R}^{d_2}$. In addition to the surface potential, there may be a random or periodic potential $V^b(x)$, which we call the "bulk" potential. The bulk potential should be stationary and ergodic with respect to shifts T_j parallel to the surface. Stationarity perpendicular to the surface is not required in the following.

This allows "interfaces" in the following sense: Let $d_1 = d - 1$, so the surface has codimension one. Thus it forms the interface between the upper

half space $V_+ = \{x; x_2 > 0\}$ and the lower half space V_-. The bulk potential V^b may then be defined by $V^b(x) = V_1(x)$ for $x_2 \geq 0$ and $= V_2(x)$ for $x_2 < 0$. Here V_1 and V_2 are random or periodic potentials on $\mathbb{R}^d$. We set $H^b = H_0 + V^b$, which we call the "bulk operator" and $H_\omega = H^b + V^s_\omega$.

We could try to define a density of states measure in the same way as in (2.1), i.e., look at

$$\lim_{L\to\infty} \frac{1}{L^d} \operatorname{tr}(\varphi(H_\omega)\chi_{\Lambda_L}). \tag{2.13}$$

It is not hard to see that this limit exists and equals

$$\mathbb{E}\left(\operatorname{tr}\left(\chi_{\Lambda_1}\varphi(H^b)\chi_{\Lambda_1}\right)\right). \tag{2.14}$$

In other words, (2.13) gives the density of states measure for the bulk operator. After all, this is not really surprising. The normalization with the volume term L^d is obviously destroying any influence of the surface potential.

So it sounds reasonable to choose a surface term like L^{d_1} as normalization and to consider

$$\lim_{L\to\infty} \frac{1}{L^{d_1}} \operatorname{tr}(\varphi(H_\omega)\chi_{\Lambda_L}). \tag{2.15}$$

However, Definition (2.15) gives a finite result only when $\operatorname{supp}\varphi \cap \sigma(H^b) = \emptyset$.

To define the density of surface states also inside the spectrum of the bulk operator, we therefore set

$$\nu_s(\varphi) = \lim_{L\to\infty} \frac{1}{L^{d_1}} \operatorname{tr}\Big(\big(\varphi(H_\omega) - \varphi(H^b)\big)\chi_{\Lambda_L}\Big). \tag{2.16}$$

Of course, it is not obvious at all that the limit (2.16) exists. In [**36, 37**] the authors proved that the limit exists for functions $\varphi \in C_0^3(\mathbb{R})$. Hence the density of surface states is defined as a distribution. The order of this distribution is at most 3. Observe that, in contrast to the density of states, the limit in (2.16) is not necessarily positive for positive φ due to the subtraction term. In fact, in the discrete case it is not hard to see that the total integral of the density of states, i.e., $\nu_s(\mathbf{1})$, is zero. Therefore, we cannot conclude that the density of surface states is a (positive) measure.

Kostrykin and Schrader [**106, 107**] proved that the density of surface states distribution is actually the derivative of a measurable locally integrable function. They do not prove that this function is of bounded variation, thus leaving the possibility that ν_s is *not* given by a measure. See also the papers [**16, 17**] by Chahrour for regularity properties of the density of surface states on the lattice.

Outside the spectrum of H^b, the distribution ν_s is positive, so that the density of surface states *is* a measure *there*. In [**87**] it was proven that below the "bulk" spectrum $\sigma(H^b)$ the density of surface states can also be defined by using (Neumann or Dirichlet) boundary conditions. We expect this to be wrong inside the bulk spectrum.

3. Lifshitz Tails

3.1. The Problem. For a periodic potential V the integrated density of states $N(E)$ behaves near the bottom E_0 of the spectrum $\sigma(H_0 + V)$ like

$$N(E) \sim C(E - E_0)^{d/2}. \tag{3.1}$$

This can be shown by explicit calculation for $V \equiv 0$ and was proved for general periodic potentials in [**81**].

On the basis of physical arguments Lifshitz [**117, 118**] predicted a completely different behavior for disordered systems, namely,

$$N(E) \sim C_1 \; e^{-C_2(E-E_0)^{-d/2}} \tag{3.2}$$

as $E \searrow E_0 > -\infty$. This behavior of $N(E)$ is called *Lifshitz behavior* or *Lifshitz tails*. The reason for this peculiar behavior is a collective phenomenon. To simplify the following heuristic argument, let us assume that $V_\omega \geq 0$ and $E_0 = 0$. To find an eigenvalue smaller than E, the potential V_ω has to be small on a rather large region in space. In fact, to have an eigenvalue at small $E > 0$, the uncertainty principle (i.e., the kinetic energy) forces the potential to be smaller than E on a set whose volume is of the order $E^{-d/2}$. That V_ω is small on a large set is a typical "large deviations event" which is very rare—in fact, its probability is exponentially small in terms of the volume of the set, i.e., its probability is of the order

$$e^{-C_2 \, E^{-d/2}} \tag{3.3}$$

which is precisely the behavior (3.2) predicted by Lifshitz. It is the aim of this section to discuss the Lifshitz behavior (3.2) of the integrated density of states as well as its extensions and limitations.

The first proof of Lifshitz behavior (for the Poisson model (1.8)) was given by Donsker and Varadhan [**33**]. They estimated the Laplace transform $\tilde{N}(t)$ for $t \to \infty$ using the Feynman–Kac representation on N (see Section 2.4). Their estimate relied on an investigation of the "Wiener sausage" and the machinery of large deviations for Markov processes developed by these authors. To obtain information about the behavior of $N(E)$ for $E \searrow 0 = E_0$ from the large t behavior of $\tilde{N}(t)$ one uses Tauberian theorems [**8, 11**]. This technique was already used by Pastur [**145, 6**] and developed in [**142, 44**] and recently in [**125**].

Donsker and Varadhan [**33**] needed in their proof of (3.2) that the single site potential f decays faster than $(1 + |x|)^{-(d+2)}$. They asked whether this condition is necessary for the result (Lifshitz tails) or just necessary for *their proof*. It was Pastur ([**146**]) who observed that, in fact, the Lifshitz asymptotic is qualitatively changed if f has long range tails, i.e., if $f(x) \sim C(1 + |x|)^{-\alpha}$ for $\alpha < d + 2$. Observe that $\alpha > d$ is necessary for the mere existence of V_ω. Pastur proved the behavior

$$N(E) \sim C_1 \; e^{-C_2(E-E_0)^{-\frac{d}{\alpha-d}}} \tag{3.4}$$

as $E \searrow E_0$ for $d < \alpha < d + 2$. We call this behavior *Pastur tails*. For a disordered system with constant magnetic field in dimension $d = 2$, Pastur tails (3.4) were found for *all* $\alpha > d = 2$ in [**10**].

These results and more observations of the last several years indicate that the asymptotics of the integrated density of states even at the bottom of the spectrum is more complicated than expected. To be more precise and following the terminology of [**148**], we can distinguish two qualitatively different behaviors in the low energy asymptotics of the integrated density of states. For short range potentials and "fat" single site distributions, the asymptotics of $N(E)$ is determined by the quantum kinetic energy as predicted by Lifshitz. Hence it is called quantum asymptotics or *quantum regime*. On the other hand, for long range potentials or "thin" single site distributions, the leading asymptotics of the integrated density of states is determined by the potential, i.e., by classical effects. This situation is called the *classical regime*.

We will discuss these phenomena in this section. We start with the short range case (quantum regime). The proof of Lifshitz tails we present here is based on spectral theoretic arguments close to Lifshitz's original heuristics (see Section 3.2).

We then discuss the long range case (classical regime) (Section 3.3) to some extent, including recent results [**86**] of single site potentials with anisotropic decay resulting in a mixed classical-quantum regime (Section 3.4).

Classical and quantum behavior of the integrated density of states and the transition between the two regimes is best understood for the Anderson model. The approach of [**125**] combines spectral theoretic and path integral methods. We will present this in Section 3.5.

Lifshitz predicted the behavior (3.1) and (3.2) not only at the bottom of the spectrum but also for any band edge of the spectrum. To distinguish these two cases, we will speak of *internal Lifshitz tails* in the latter case. Investigating Lifshitz behavior at internal band edges turns out to be much more complicated than at the bottom of the spectrum. In fact, already the investigation of periodic potentials at internal band edges is extremely complicated. We will discuss internal Lifshitz tails (following [**97, 104, 105**]) in Section 3.6.

Finally, we will look at random Schrödinger operators with magnetic fields in Section 3.8.

3.2. Lifshitz Tails: Quantum Case.

3.2.1. *Statement of the main result.* The aim of this subsection is a proof of Lifshitz behavior close to his original heuristics and without heavy machinery. We will prove the quantum asymptotics in (3.2) for short range single site potentials and "fat" single site distributions. We will make no attempt to reach high generality but rather emphasize the strategy of the proof.

As before, we consider random alloy-type potentials of the form

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^d} q_i(\omega)\, f(x-i). \tag{3.5}$$

We assume that the random variables q_i are independent and identically distributed with a common probability distribution P_0. We suppose that the support of P_0 is compact and contains at least two points.

As always, we also suppose that the single site potential f is non-negative, bounded and decays at infinity as fast as $|\,x|^{-(d+\varepsilon)}$. The technique we are going to present allows us to treat local singularities of f. (We refer to [**80, 86**] for details.)

To ensure Lifshitz tails in the sense of (3.2), we need two conditions:

Assumption 1: Define $q_{min} = \inf \operatorname{supp}(P_0)$. We assume that

$$P_0\big([q_{\min}, q_{\min}+\varepsilon)\big) \;\geq\; C\,\varepsilon^N \tag{3.6}$$

for some C, N and all $\varepsilon > 0$ small.

Condition (3.6) means that the distribution P_0 is "fat" at the bottom of its support. Note that this condition is, in particular, satisfied if P_0 has an atom at $q_{\min}$, i.e., if $P_0(\{q_{\min}\}) > 0$.

The second condition we need is precisely the "short range" condition already encountered by Donsker and Varadhan [**33**].

Assumption 2:

$$f(x) \leq C\,(1+|\,x|)^{-(d+2)}. \tag{3.7}$$

We are ready to formulate the main result of this subsection.

THEOREM 3.1. *If Assumptions (1) and (2) are satisfied, we have*

$$\lim_{E\searrow E_0} \frac{\ln(-\ln N(E))}{\ln(E-E_0)} = -\frac{d}{2}. \tag{3.8}$$

Observe that equation (3.8) is a weak form of Lifshitz's original conjecture (3.2). In their work [**33**], Donsker and Varadhan proved the stronger form for the Poisson potential

$$\lim_{E\searrow 0} \frac{\ln N(E)}{E^{-d/2}} = -C_d \tag{3.9}$$

where C_d is a (computable) positive constant.

Both the short range condition (Assumption 2) and the fatness condition (Assumption 1) turn out to be necessary for the above result, as we will see later. For example, if the single site potential f decays substantially slower than required in the short range condition the integrated density of states decays *faster* than in the (3.8).

We define the *Lifshitz exponent* γ by

$$\gamma = \lim_{E\searrow E_0} \frac{\ln(-\ln N(E))}{\ln(E-E_0)} \tag{3.10}$$

whenever this limit exists. With this notation we may rephrase (3.8) as $\gamma = -d/2$. The Lifshitz exponent for periodic potentials is 0.

3.2.2. *Strategy of the proof.* The proof of Theorem 3.1 consists of an upper and a lower bound. The next subsection will provide us with the tools we need for these bounds.

It will turn out that the bounds are easier and more natural for positive random potentials. Therefore, we will split the random potential in a periodic and a positive random part

$$\begin{aligned} V_\omega(x) &= \sum_{i\in\mathbb{Z}^d} q_{\min}\, f(x-i) + \sum_{i\in\mathbb{Z}^d} (q_i(\omega) - q_{\min})\, f(x-i) \\ &= V_{\mathrm{per}}(x) + \tilde{V}_\omega(x). \end{aligned} \tag{3.11}$$

We will subsume the periodic potential under the kinetic energy and denote the positive random potential $\tilde{V}_\omega$ in a slight abuse of notation again by V_ω. Thus we have

$$H_\omega = H_1 + V_\omega \tag{3.12}$$

with $H_1 = H_0 + V_{\mathrm{per}}$ a Hamiltonian with periodic "background" potential V_{per} and

$$V_\omega = \sum_{i\in\mathbb{Z}^d} q_i(\omega)\, f(x-i) \tag{3.13}$$

where the independent $q_i \geq 0$ have a common probability distribution P_0 with $0 = \inf(\mathrm{supp} P_0)$.

For the upper bound below, we need information about the two lowest eigenvalues of H_1 restricted to a box. If $V_{\mathrm{per}} \equiv 0$, these eigenvalues can be computed explicitly. However, if $V_{\mathrm{per}} \not\equiv 0$, we need a careful analysis of periodic operators. This was done in [**81**] and [**129**]. Here we restrict ourselves to the case $V_{\mathrm{per}} \equiv 0$, avoiding some technical complications. Note that this implies $E_0 = \inf(\sigma(H_\omega)) = 0$. We refer the reader to the papers [**80**] and [**129**] for the general case. We also remark that [**86**] contains an extension of the approach presented here that works for Poisson potentials and various other potentials as well.

3.2.3. *The Dirichlet–Neumann bracketing.* The first step in the proof is to bound the integrated density of states from above and from below using the Dirichlet–Neumann bracketing as in Corollary 2.7. We have

$$\frac{1}{|\Lambda_L|}\, \mathbb{E}\left(N(H^D_{\omega\,\Lambda_L}, E)\right) \leq N(E) \leq \frac{1}{|\Lambda_L|}\, \mathbb{E}\left(N(H^N_{\omega\,\Lambda_L}, E)\right). \tag{3.14}$$

The side length L of the cube Λ_L will be chosen later in an E-dependent way when we send E to E_0. We estimate the right-hand side of (3.14) by

$$\begin{aligned}\mathbb{E}(N(H^N_{\omega\,\Lambda_L}, E) &= \int N(H^N_{\omega\,\Lambda_L}, E)\; d\,\mathbb{P} \\ &= \int_{E_1(H^N_{\omega\,\Lambda_L})\le E} N(H^N_{\omega\,\Lambda_L}, E)\; d\,\mathbb{P} + \int_{E_1(H^N_{\omega\,\Lambda_L})>E} N(H^N_{\omega\,\Lambda_L}, E)\; d\,\mathbb{P} \\ &\le \mathbb{P}\,(E_1(H^N_{\omega\,\Lambda_L}) \le E)\; N(H_0{}^N_{\Lambda_L}, E\,).\end{aligned}$$

With $N(H_0{}^N_{\Lambda_L}, E\,) \le (C_1 + C_2E)^{d/2}|\Lambda|$ following from Weyl asymptotics, we get for $0 \le E \le 1$ the estimate

$$\frac{1}{|\Lambda_L|}\,\mathbb{P}\left(E_1(H^D_{\omega\,\Lambda_L}) \le E\right) \le N(E) \le C\,\mathbb{P}\left(E_1(H^N_{\omega\,\Lambda_L}) \le E\right). \tag{3.15}$$

The problem now is to find upper and lower bounds such that after taking the double logarithm, the left- and the right-hand side of (3.15) coincide asymptotically. In general the upper bounds are more difficult than the lower bounds. To prove the lower bound we only have to "guess" a good test function, whereas for the upper bounds, one has to prove that all eigenfunctions for energies in $[0, E)$ roughly behave the same way.

It is an astonishing fact that the lower bound from (3.15) in all known cases leads to the *asymptotically correct* behavior of the integrated density of states, a fact emphasized by Pastur.

3.2.4. *The lower bound.* For simplicity we restrict ourselves to single site potentials f with $\operatorname{supp} f \subset \Lambda_{\frac12}$ so that $f(\cdot - i)$ and $f(\cdot - j)$ do not overlap for $i \ne j$. The necessary changes for the general case will become clear when we discuss long range potentials f.

By the Neumann–Dirichlet bracketing in (3.15), we have for arbitrary L and any $\psi \in D(\Delta^D_{\Lambda_L})$ with $||\psi||_{L^2(\Lambda_L)} = 1$,

$$\begin{aligned}N(E) &\ge |\Lambda_L|^{-1}\mathbb{P}\left(E_1(H_{\omega\,\Lambda_L}{}^D) < E\right) \\ &\ge |\Lambda_L|^{-1}\mathbb{P}\left(\langle\psi, H_\omega\psi\rangle_{L^2(\Lambda_L)} < E\right).\end{aligned} \tag{3.16}$$

A natural choice of ψ for (3.16) seems to be the ground state ψ_0 of $-\Delta^N_{\Lambda_L}$, $\psi_0(x) \equiv |\Lambda_L|^{-\frac12}$. Unfortunately, this function does not obey Dirichlet boundary conditions and is therefore not admissible for (3.16).

This problem can be circumvented by multiplying ψ_0 by a function which is zero at the boundary of Λ_0. To do so, let us take $\chi \in C^\infty(\mathbb{R}^d)$, $\operatorname{supp}\chi \subset \Lambda_1$, $\chi(x) = 1$ on $\Lambda_{\frac12}$ and $0 \le \chi(x) \le 1$. We set $\chi_L(x) = \chi(\frac{x}{L})$ and $\psi_L(x) = \chi_L(x)\psi_0(x)$. Then $\psi_L \in D(\Delta^D_{\Lambda_L})$, $||\psi_L|| \ge \frac12$ and

$$\begin{aligned}\langle\psi_L, H_\omega\psi_L\rangle &\le \langle\psi_0, H_\omega\psi_0\rangle + CL^{-2} \\ &= |\Lambda_L|^{-1}\int_{\Lambda_L} V_\omega(x)dx + CL^{-2}.\end{aligned}$$

Note that the "error term" L^{-2} is due to the influence of the kinetic energy (a second-order differential operator). Inserting in (3.16) we get

$$\begin{aligned} N(E) &\geq |\Lambda_L|^{-1}\, \mathbb{P}\left(|\Lambda_L|^{-1} \int_{\Lambda_L} V_\omega(x) < E - CL^{-2} \right) \\ &\geq |\Lambda_L|^{-1}\, \mathbb{P}\Big(|\Lambda_L|^{-1} \parallel f \parallel_1 \sum_{i\in\Lambda_L} q_i(\omega) < E - CL^{-2} \Big). \end{aligned} \tag{3.17}$$

In principle, we can choose L as we like. However, if $E < CL^{-2}$ estimate (3.17) becomes useless. So it seems reasonable to choose $L = \beta E^{-\frac{1}{2}}$ and we obtain

$$\begin{aligned} (3.17) &\geq |\Lambda_L|^{-1}\, \mathbb{P}\Big(|\Lambda_L|^{-1} \sum_{i\in\Lambda_L} q_i(\omega) < \tilde{C} E \Big) \\ &\geq |\Lambda_L|^{-1}\, \mathbb{P}\left(q_0 < \tilde{C}\, E \right)^{L^d} \\ &\geq C_1 E^{-\frac{d}{2}} \left(C\, E^N \right)^{C_2 E^{-\frac{d}{2}}}. \end{aligned}$$

Thus we conclude

$$\lim_{E\searrow 0} \frac{\ln(-\ln(N(E)))}{\ln E} \geq -\frac{d}{2}. \tag{3.18}$$

3.2.5. *The upper bound.* The strategy to prove the upper bounds of $\mathbb{P}\left(E_1(H^N_{\omega_{\Lambda_L}}) < E\right)$ in (3.15) can be divided in two parts. The first step is to find a *lower* bound for $E_1(H^D_{\omega_{\Lambda_L}})$ in such a way that it is possible to control the influence of the random potential. This is done by an application of Temple's inequality. The second step is to balance between the size of Λ_L and the probability of a random potential such that most of the potential values are small.

We start by stating Temple's inequality for the reader's convenience. A proof can be found, e.g., in [**150**].

THEOREM 3.2 (Temple's inequality). *Suppose H is a self-adjoint operator, bounded below which has discrete spectrum and denote by $E_n(H), n = 1, 2, \ldots$ its eigenvalues (in increasing order, counting multiplicity). If $\mu \leq E_2(H)$ and $\psi \in D(H)$ with $\|\psi\| = 1$ satisfying $\langle \psi, H\,\psi\rangle < \mu$, then*

$$E_1(H) \geq \langle \psi, H\psi\rangle - \frac{\langle \psi, H^2\,\psi\rangle - \langle \psi, H\psi\rangle^2}{\mu - \langle \psi, H\,\psi\rangle}.$$

To apply Temple's inequality, we set $E_2(-\Delta^N_{\Lambda_L}) := \mu \leq E_2(H_{\omega}{}^N_{\Lambda_L})$. Note that by direct computation, $E_1(-\Delta^N_{\Lambda_L}) = 0 = E_0$ and $E_2(-\Delta^N_{\Lambda_L}) \sim L^{-2}$. These facts require a careful analysis if there is a periodic background potential as in (3.11) and (3.12); see [**81**].

Next we need a good approximation ψ of the ground state of $H_{\omega}{}^N_{\Lambda_L}$. This is done by choosing ψ to be the ground state of $-\Delta^N_{\Lambda_L}$, which is intuitively

close to the correct ground state for small E. The function ψ is given by $\psi(x) = |\Lambda_L|^{-1/2}$.

To apply Temple's inequality we have to ensure that with the above choice, $\langle \psi, H\,\psi\rangle < \mu \approx c\,L^{-2}$. We force this to happen by changing the coupling constants q_i to $\tilde{q}_i = \min(q_i(\omega), \alpha L^{-2})$ with a suitable $\alpha > 0$, small enough. If $\tilde{H}$ denotes the corresponding operator, we have $E_1(H) \geq E_1(\tilde{H})$. An application of Temple's inequality to $\tilde{H}$ and an elementary calculation yield the following lemma.

Lemma 3.3.

$$E_1(H_{\omega\,\Lambda_L}^{\,N}) \geq \frac{1}{2}\,\frac{1}{|\Lambda_L|}\,\sum_{i\in\Lambda_L}\tilde{q}_i(\omega). \tag{3.19}$$

A consequence of the lemma above is the intuitively convincing estimate

$$\mathbb{P}\left(E_1(H_{\omega\,\Lambda_L}^{\,N}) < E\right) \leq \mathbb{P}\left(\frac{1}{|\Lambda_L|}\sum_{i\in\Lambda_L}\tilde{q}_i(\omega) \leq 2\,E\right) \tag{3.20}$$

$$\leq \mathbb{P}\left(\frac{1}{L^d}\sum_{|i|_\infty \leq L/2}\tilde{q}_i(\omega) \leq 2\,E\right). \tag{3.21}$$

The expression (3.21) for E small very much resembles a large deviation probability which would lead to a bound exponentially small in the volume term L^d. At first sight, Cramer's theorem, a result of the theory of large deviation, seems to be applicable (see, e.g., [**31, 32, 58**]).

However, there is a complication here: To obtain a large deviation event in (3.21) we need that $\mathbb{E}(\tilde{q}_i) < E$. Thus, if we set $L = L(E) = \beta E^{-1/2}$ with $\beta > 0$ small, the event (3.21) is, indeed, a large deviation event and we obtain the following bound.

Lemma 3.4.

$$\mathbb{P}\left(\frac{1}{|\Lambda_L|}\sum_{i\in\Lambda_L}\tilde{q}_i(\omega) \leq E\right) \leq C_1\; e^{-\,C_2\,L^d}$$

for E close enough to zero and $L \leq \beta\,E^{-1/2}$.

Combining the results above, (3.15) and (3.20), we have proven

$$N(E) \leq C_1^{-C_2 E^{-d/2}}.$$

3.2.6. *Final remarks.* The idea of using Neumann–Dirichlet bracketing to prove Lifshitz tails first appeared in [**77**]. It was carried over to the discrete Anderson model by Simon [**156**] who streamlined it at the same time. The proof was extended to more general alloy-type potentials by Kirsch and Simon [**80**], who still needed reflection symmetry of f. Mezincescu [**129**] modified the upper bound by introducing other boundary conditions to get rid of this extra assumption. We refer to [**86**] for a rather general proof using these techniques.

3.3. Long Range Single Site Potentials: A "Classical" Case. In this section we turn to an example of classical behavior of the integrated density of states near $0 = \inf(\sigma(H_\omega))$, in the sense of Section 3.1, namely, to long range single site potential f.

The upper bound on $N(E)$ is easier than for the short range case. While there is a subtle interplay between the kinetic energy and the potential in the short range case ($f(x) \leq |x|^{-(d+2)}$), it is the potential energy alone that determines the leading behavior of $N(E)$ ($E \searrow 0$) in the long range case.

Assumption: In this section we suppose that

$$\frac{c}{(1+|x|)^\alpha} \leq f(x) \leq \frac{C}{(1+|x|)^\alpha}. \tag{3.22}$$

THEOREM 3.5. *Assume* (3.6). *If* (3.22) *holds for an* α *with* $d < \alpha < d+2$, *then*

$$\lim_{E \searrow E_0} \frac{\ln(-\ln N(E))}{\ln E} = -\frac{d}{\alpha - d}. \tag{3.23}$$

In the terminology of (3.10), Theorem 3.5 states that the Lifshitz exponent for the long range case ($\alpha < d+2$) is $d/(\alpha - d)$.

PROOF. To simplify the argument, we assume as in the short range case, there is no periodic background potential and $q_{\min} = 0$. Consequently, $E_0 = 0$. We start with the upper bound and estimate

$$E_1(H_{\omega \Lambda_1}^N) \geq \inf_{x \in \Lambda_1} \sum_{i \in \mathbb{Z}^d} q_i \frac{c}{(x+|i|)^\alpha}.$$

Hence

$$\begin{aligned} N(E) &\leq C_1 \, \mathbb{P}\left(E_1(H_{\omega \Lambda_1}^N) < E\right) \\ &\leq C_1 \, \mathbb{P}\left(\sum_{i \in \mathbb{Z}^d} q_i \frac{C_2}{(1+|i|)^\alpha} < E \right) \\ &\leq C_1 \, \mathbb{P}\left(\sum_{|i| \leq L} q_i \frac{C'}{L^\alpha} < E \right) \\ &\leq C_1 \, \mathbb{P}\left(\frac{1}{L^d} \sum_{|i| \leq L} q_i < C_3 E L^{\alpha - d} \right). \end{aligned} \tag{3.24}$$

We choose $L = \frac{\beta}{C_3} E^{-\frac{1}{\alpha - d}}$ (β small). Hence

$$(3.24) \leq C_1 \, \mathbb{P}\left(\frac{1}{L^d} \sum_{|i| \leq L} q_i \leq \beta \right).$$

If $\beta < \frac{1}{2} \mathbb{E}(q_0)$, standard large deviation theory gives

$$\mathbb{P}\left(\frac{1}{L^d} \sum_{|i| \leq L} q_i \leq \beta \right) < e^{-CL^d} = e^{-\tilde{C} E^{-\frac{d}{\alpha - d}}}.$$

We turn to the lower bound. As in the proof of the lower bound (3.17) in the previous section,

$$\begin{aligned} N(E) &\geq \frac{1}{|\Lambda_L|}\mathbb{P}\left(\frac{1}{|\Lambda_L|}\int_{\Lambda_L} V_\omega(x)dx < E - CL^{-2}\right) \\ &\geq \frac{1}{|\Lambda_L|}\mathbb{P}\Big(\sum_{i\in\mathbb{Z}^d} q_i \frac{1}{|\Lambda_L|}\int_{\Lambda_L} f(x-i)dx < E - CL^{-2}\Big). \end{aligned}$$

Due to the long range tails of f, we cannot ignore the summands with $|i|$ large. Instead, we estimate

$$\begin{aligned} &\sum_{i\in\mathbb{Z}^d} q_i \,\frac{1}{|\Lambda_L|}\int_{\Lambda_L} f(x-i)\,dx \\ &\qquad \leq \frac{1}{|\Lambda_L|}\sum_{|i|_\infty\leq 2L} q_i \int f(y)\,dy + \sum_{|i|_\infty > 2L} q_i \frac{1}{|\Lambda_L|}\int_{\Lambda_L} f(x-i)\,dx \\ &\qquad \leq \frac{C_3}{|\Lambda_{2L}|}\sum_{|i|_\infty\leq 2L} q_i + q_{\max}\sum_{|i|_\infty > 2L}\frac{1}{|\Lambda_L|}\int_{\Lambda_L} f(x-i)\,dx, \end{aligned}$$

where $q_{\max} = \sup(\mathrm{supp} P_0)$, P_0 being the distribution of the q_i. We estimate

$$\begin{aligned} \sum_{|i|>2L}\frac{1}{|\Lambda_L|}\int_{\Lambda_L} f(x-i)dx &\leq C_4 \sum_{|i|>2L}\frac{1}{|\Lambda_L|}\int_{\Lambda_L}\frac{1}{|x-i|^\alpha}dx \\ &\leq C_5\sum_{|i|>L}\frac{1}{|i|^\alpha} \\ &\leq C_6\frac{1}{L^{\alpha-d}}. \end{aligned}$$

Thus we obtain

$$N(E) \geq \frac{1}{|\Lambda_L|}\mathbb{P}\Big(\frac{1}{|\Lambda_{2L}|}\sum_{|i|\leq 2L} q_i \leq C_7 E - C_8 L^{-2} - C_9 L^{(\alpha-d)}\Big). \tag{3.25}$$

As the derivation shows, the L^{-2} term comes from manipulating the kinetic energy, while the $L^{-(\alpha-d)}$ term is due to the potential energy. Note that for $\alpha > d+2$, the term L^{-2} (kinetic energy contribution) dominates in (3.25). In this case we can therefore redo the estimates of Section 3.2.4 and obtain a lower bound as we got there. However, for $\alpha < d+2$, the term $L^{-(\alpha-d)}$ wins out in (3.25). Remember, this term is due to the potential energy distribution. We obtain

$$N(E) \geq \frac{1}{|\Lambda_L|}\mathbb{P}\Big(\frac{1}{|\Lambda_{2L}|}\sum_{|i|\leq 2L} q_i \leq C_7 E - C_{10} L^{-(\alpha-d)}\Big).$$

This time, E has to be bigger than $L^{-(\alpha-d)}$; more precisely, $E \geq C_{11}L^{-(\alpha-d)}$. Hence $L = C_{12}E^{-\frac{1}{\alpha-d}}$ so

$$\begin{aligned} N(E) &\geq \frac{1}{|\Lambda_L|}\mathbb{P}\left(q_i = 0 \text{ for } |i| \leq 2L\right) \\ &\geq \frac{1}{|\Lambda_L|}e^{-C_{13}L^d} \\ &\geq C_{14}E^{\frac{d}{\alpha-d}}\ e^{-C_{15}E^{-\frac{d}{\alpha-d}}}. \end{aligned}$$

□

3.4. Anisotropic Single Site Potentials. Recently, Theorems 3.1 and 3.5 were generalized to single site potentials f decaying in an anisotropic way at infinity ([**86**]). Let us write $x \in \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ as $x = (x_1, x_2)$, $x_1 \in \mathbb{R}^{d_1}$, $x_2 \in \mathbb{R}^{d_2}$ and suppose that

$$\frac{a}{|x_1|^{\alpha_1} + |x_2|^{\alpha_2}} \leq f(x) \leq \frac{b}{|x_1|^{\alpha_1} + |x_2|^{\alpha_2}} \tag{3.26}$$

for $|x_1|, |x_2| \geq 1$, and define V_ω in the usual way

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^d} q_i f(x-i). \tag{3.27}$$

Let us define $\gamma_i = \frac{d_i}{\alpha_i}$ and $\gamma = \gamma_1 + \gamma_2$. Then the sum in (3.27) converges (absolutely) if $\gamma < 1$.

In [**86**] the authors prove that there is Lifshitz behavior of $N(E)$ for potential as in (3.26) and (3.27) in the sense that the Lifshitz exponent η, defined by

$$\eta = \lim_{E\searrow E_0} \frac{\ln|\ln(N(E))|}{\ln E}$$

exists ($E_0 = \inf\sigma(H_\omega)$). η depends on the exponents α_i, of course. If both

$$\frac{\gamma_1}{1-\gamma} \leq \frac{d_1}{2} \quad \text{and} \quad \frac{\gamma_2}{1-\gamma} \leq \frac{d_2}{2} \tag{3.28}$$

we obtain the "quantum" exponent:

$$\eta = -\frac{d}{2}.$$

Observe that (3.28) reduce to the condition $\alpha \leq d+2$ for the isotropic case $\alpha_1 = \alpha_2 = \alpha$. If

$$\frac{\gamma_1}{1-\gamma} > \frac{d_1}{2} \quad \text{and} \quad \frac{\gamma_2}{1-\gamma} > \frac{d_2}{2} \tag{3.29}$$

we are in the "classical" case both in the d_1- and d_2-directions. Then

$$\eta = -\frac{\gamma}{1-\gamma}.$$

The third case

$$\frac{\gamma_1}{1-\gamma} \leq \frac{d_1}{2} \quad \text{and} \quad \frac{\gamma_2}{1-\gamma} > \frac{d_2}{2} \tag{3.30}$$

is new compared to the isotropic case. It is, in a sense, a mixed quantum-classical case. The Lifshitz exponent is given by

$$\eta = -\frac{d_1}{2} - \frac{\gamma_2}{1-\gamma}.$$

We note that the d_1-direction and the d_2-direction "influence each other" in a rather sophisticated way. In [**86**] these results are proved for alloy-type potentials as well as for Poisson (and related) models. We summarize:

THEOREM 3.6 ([**86**]). *Suppose* (3.6) *and* (3.26) *hold. Set* $\gamma_i = d_i/\alpha_i$ *and* $\gamma = \gamma_1 + \gamma_2$. *Then the Lifshitz exponent* η *is given by*

$$\eta = -\max\left\{\frac{d_1}{2}, \frac{\gamma_1}{1-\gamma}\right\} - \max\left\{\frac{d_2}{2}, \frac{\gamma_2}{1-\gamma}\right\}. \tag{3.31}$$

3.5. Path Integral Methods and the Transition Between Quantum and Classical Regime with Respect to the Single Site Measure. We start this section with two observations which indicate that the asymptotic behavior of the integrated density of states depends qualitatively on the distribution P_0 of the q_i.

Let us first assume that the "fatness" condition (3.6) is satisfied in the strongest form, namely, $\inf \operatorname{supp}(P_0) = 0$ and $P(q_0 = 0) = a > 0$. An inspection of the proofs in Section 3.2 shows that for this case we have actually proven

$$\liminf \frac{\ln N(E)}{(E-E_0)^{-d/2}} \geq -C_1, \tag{3.32}$$

$$\limsup \frac{\ln N(E)}{(E-E_0)^{-d/2}} \leq -C_2 \tag{3.33}$$

with $C_1, C_2 > 0$. If instead $P(q_0 = 0) = 0$ (but still $P(q_0 > \epsilon) \geq B\epsilon^n$) the lower bound requires a logarithmic correction

$$\liminf \frac{\ln N(E)}{(E-E_0)^{-d/2}|\ln(E-E_0)|} \geq -C_1. \tag{3.34}$$

The second observation concerns unbounded single site measures P_0. In [**98**] it is proved that both the classical and the quantum regime can occur for the discrete, unbounded Anderson model and more general matrix operators. Depending on the single site measure, collective phenomena may occur similar to those we encountered above. In other situations, the single site measure alone determines the behavior of the integrated density of states.

It seems difficult to understand the mechanisms causing the transition from quantum to classical regime with respect to the single site measure by using the spectral analytic approach close to Lifshitz's original intuition.

The first approach to prove Lifshitz tails is based on the Donsker–Varadhan technique (see also Section 3.1). This method to compute the Laplace transform of the integrated density of states in the limit $t \to \infty$

is a far-reaching generalization of the Laplace method known from classical analysis. The starting point is the path integral representation

$$\tilde{N}(t) = \mathbb{E} \times \mathbb{E}_{0,0}^{t,0}\big[e^{-\int_0^t V_\omega(b(s))ds}\big].$$

The Donsker–Varadhan technique is based on a large deviation principle satisfied by the product probability measure $d\mathbb{P} \times d\mathbb{P}_{0,0}^{t,0}$ combining the random potential and the Brownian motion. In an informal sense, it makes it possible to quantize the asymptotic probability of a Brownian particle to stay most of its lifetime in a pocket with a favorable configuration of potential values. Using the large deviation principle, one can balance between favorable configurations and their small probability by applying Varadhan's lemma. Last but not least, given the large time asymptotics of the Laplace transform $\tilde{N}(t)$, one can reconstruct the Lifshitz tail behavior using Tauber theory.

The Donsker–Varadhan technique was worked out by Nakao [**142**] for the Poisson model with $f \geq 0$. He proved

$$\lim_{E \searrow 0} \frac{\ln N(E)}{E^{-d/2}} = -C_d$$

where C_d is a (computable) positive constant. In the 1990's Lifshitz asymptotics became a starting point for stochastic analysis of diffusion in random media. We mention the work of Sznitman (see, e.g, [**167, 168, 169, 170**]) in the continuous case, especially for Poisson potentials, and in the discrete context the moment analysis for the so-called parabolic Anderson model (PAM) (see, e.g., [**7, 48, 49, 50**]). Here Brownian motion has to be replaced by the continuous time Markov chain generated by the discrete Laplacian.

The phenomenology described at the beginning of this subsection was also observed in the moment analysis of the parabolic Anderson model starting in [**48**]. As we will see, the case of the double exponential distribution discussed in [**49**] can be interpreted as the borderline between the quantum and the classical regime. The paper [**7**] clarified the discrepancy between the lower and the upper bounds in (3.32), (3.33) and (3.34). Still, a general principle explaining the transition from quantum to classical regime with respect to the single site measure was not formulated.

We want to systemize the phenomenology discussed above in the following theorem taken from [**125**]. To combine the bounded and the unbounded case, we assume that the cumulant generating function is finite, e.g.,

$$G(t) := \log \mathbb{E}\left(\exp(-tV_\omega(0))\right) < \infty \tag{3.35}$$

for all $t \geq 0$. Furthermore, we apply the Legendre transformation to define the rate function

$$I(E) := \sup_{t>0}[Et - G(t)] \tag{3.36}$$

and for $t, \lambda > 0$, we set

$$S(\lambda, t) := (\lambda t)^{-1} G(\lambda t) - t^{-1} G(t). \tag{3.37}$$

Informally the scale function $S(\lambda, t)$ measures the change of the cumulant generating function after rescaling the time.

THEOREM 3.7. *We consider the discrete Anderson tight binding operator $h_\omega = h_0 + V_\omega$ and set $E_0 = \inf \sigma(h_\omega)$. Suppose $G(t) < \infty$ for all $t > 0$. Then we distinguish the following four cases:*

(i) *Let $S(\lambda, t) \sim c\,(\lambda^\rho - 1)t^\rho$ with $c, \rho > 0$. Then the IDS behaves in the limit $E \to E_0 = -\infty$ like*

$$\log(N(E)) = -I(E + 2d + o(1))(1 + o(1)).$$

(ii) *In the case $S(\lambda, t) \sim c \log(\lambda)$ with $\rho = 0$ and $c > 0$, we have in the limit $E \to E_0 = -\infty$,*

$$-KI(E + C_2)(1 + o(1)) \leq \log(N(E)) \leq -I(E + C_1)(1 + o(1))$$

with $K > 0$,

$$C_1 = -2\sin^2\left(\frac{\pi}{2}\,\frac{1}{c^{-1/2} + 1}\right) + \frac{1}{4}\,c\log(c)$$

and

$$C_2 = K(d)\min\left[-c + c\log(c^{-1}), \max\left[1 - 4\exp(-Kc), \frac{a(d)}{4}\right]\right].$$

(iii) *In the case $S(\lambda, t) \sim c\,(1 - \lambda^\rho)t^\rho$ with $-1 < \rho < 0$ and $c > 0$, the IDS behaves in the limit $E \searrow E_0 = 0$ like*

$$\begin{aligned} -K_1 E^{-1/2(d - 2\rho^{-1}(\rho+1))}(1 + o(1)) &\leq \log(N(E)) \\ &\leq -K_2 E^{-1/2(d - 2\rho^{-1}(\rho+1))}(1 + o(1)). \end{aligned}$$

(iv) *In the case $S(\lambda, t) \sim -c\,(\lambda t)^{-1}\log(t)$ with $c > 0$, we have in the limit $E \searrow E_0 = 0$,*

$$\begin{aligned} -K_1 E^{-d/2}\log(E)(1 + o(1)) &\leq \log(N(E)) \\ &\leq -K_2 E^{-d/2}\log(E)(1 + o(1)). \end{aligned}$$

The scaling assumption $S(\lambda, t) \sim c\,(\lambda^\rho - 1)t^\rho$ with $c, \rho > 0$ in the first case corresponds to "fat" unbounded single site distributions; the behavior of the integrated density of states is classical. The second case represents the double exponential case, while in the third situation the single site distribution is bounded, but very thin. The fourth case corresponds to relatively fat single site distributions studied in [**74**].

Although, by now, there are results covering a lot of possible single site distributions, there seems to be no systematic approach known to explain this phenomenology. Furthermore there exists two relatively different approaches as discussed above. A first step to combine the functional analytic and the path integral approach as well as to systemize the known results with respect to the single site distribution seem to be [**125, 126**]. In contrast to the direct analysis of the operator H_ω in Section 3.2, but in analogy to

the path integral methods, one is interested in the large time behavior of the semigroup $\exp(-h_\omega t)$. The Lifshitz asymptotics follows by an application of (modified) Tauber theorems [**127**].

The first step of the argument in [**125**] is to restrict h_ω to a (time-dependent) box $\Lambda = \Lambda_t(0)$ by introducing discrete Dirichlet boundary conditions and to approximate $\mathbb{E}[\exp(-h_\omega t)(0,0)]$ in the limit $t \to \infty$ by

$$\mathbb{E}[\exp(-E_1(h_\omega^\Lambda)t)] = \mathbb{E}\Big[\sup_{p\in M_1(\Lambda)} \exp\big(-t\ \big[\big(\sqrt{p}|h_0^\Lambda\sqrt{p}\big) + \big(\sqrt{p}|V(\omega)\sqrt{p}\big)\big]\big)\Big]. \tag{3.38}$$

Here $E_1(h_\omega^\Lambda) = \inf\sigma(h_\omega^\Lambda)$ is the principal eigenvalue of h_ω^Λ and $M_1(\Lambda)$ is the set of probability measures on Λ. Equation (3.38) is a consequence of the min-max principle and the nonnegativity of the ground state. It is the starting point to find upper and lower bounds of the Laplace transform of the IDS.

To illustrate the central effect explaining the transition from the quantum mechanical to the classical regime, we want to sketch the very elementary proof of the lower bounds starting from (3.38). The first step is to interchange the expectation value and the supremum

$$\begin{aligned}
\mathbb{E}\left[\exp(-E_1(h_\omega^\Lambda)t)\right] &= \mathbb{E}\Big[\sup_{p\in M_1(\Lambda)} \exp\big(-t\ \big[\big(\sqrt{p}|h_0^\Lambda\sqrt{p}\big) + \big(\sqrt{p}|V(\omega)\sqrt{p}\big)\big]\big)\Big] \\
&\geq \sup_{p\in M_1(\Lambda)} \exp\big(-t\ \big(\sqrt{p}|h_0^\Lambda\sqrt{p}\big)\big)\, \mathbb{E}\Big[\exp\Big(-\sum_{x\in\Lambda} p(x)V_\omega(x)t\Big)\Big] \\
&= \sup_{p\in M_1(\Lambda)} \exp\Big(-t\ \big(\sqrt{p}|h_0^\Lambda\sqrt{p}\big) + \sum_{x\in\Lambda} G(p(x)t)\Big).
\end{aligned} \tag{3.39}$$

The second step is to define a subset $D \subset M_1(\Lambda)$ of relatively uniform probability distributions concentrated on a subvolume of Λ. With the side length L of Λ, we set $1 \leq l \leq L$ and $\Lambda_l := \{x \in \mathbb{Z}^d : |x|_\infty \leq l\}$. The ground state of the discrete Laplacian $h_0^{\Lambda_l}$ restricted to Λ_l with Dirichlet boundary conditions is given by

$$\phi_l \colon \Lambda_l \to [0,\infty), \tag{3.40}$$

$$\phi_l = \prod_{j=1}^d \left(\frac{2}{l+1}\right)^{1/2} \sin\left(\frac{x_j\pi}{l+1}\right) \tag{3.41}$$

and the corresponding principal eigenvalue of $h_0^{\Lambda_l}$ is

$$E_1(h_0^{\Lambda_l}) = 2d\left(1 - \cos\left(\frac{\pi}{l+1}\right)\right). \tag{3.42}$$

The subset $D \subset M_1(\Lambda)$ is then defined by

$$D := \{\phi_l^2 : 1 \leq l \leq L\}. \tag{3.43}$$

Restricting the estimate (3.39) to D, we get

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(-t\ E_1\left(h_\omega^\Lambda\right)\right)\right)\\
&\qquad\geq\ \exp\Bigg(G(t)+t\sup_{p\in M_1(\Lambda)}\Bigg(-(\sqrt{p}|h_0^\Lambda\sqrt{p})\ +\ \sum_{x\in\Lambda}p(x)S(p(x),t)\Bigg)\Bigg)\\
&\qquad\geq\ \exp\Bigg(G(t)+t\sup_{p\in D}\Bigg(-(\sqrt{p}|h_0^\Lambda\sqrt{p})\ +\ \sum_{x\in\Lambda}p(x)S(p(x),t)\Bigg)\Bigg).
\end{aligned}$$

Using the definition of $S(\lambda,t)$, the convexity of the cumulant generating function $G(t)$ and the Jensen inequality, we can estimate for $p\in M_1(\Lambda_l)\subset M_1(\Lambda)$,

$$\begin{aligned}
\sum_{x\in\Lambda}p(x)S(p(x),t)\ &=\ \sum_{x\in\Lambda_l}p(x)\left(\frac{G(p(x)t)}{p(x)t}-\frac{G(t)}{t}\right)\\
&=t^{-1}l^d\left(l^{-d}\sum_{x\in\Lambda_l}G(p(x)t)\right)-t^{-1}G(t)\\
&\geq t^{-1}l^dG\left(l^{-d}t\sum_{x\in\Lambda_l}p(x)\right)-t^{-1}G(t)\\
&=\frac{G(l^{-d}t)}{l^{-d}t}-\frac{G(t)}{t}\\
&=S(l^{-d},t).
\end{aligned}$$

So the uniform distribution on Λ_l minimizes $\sum_{x\in\Lambda}p(x)S(p(x),t)$ with respect to the variation over $p\in M_1(\Lambda_l)$. We have

$$\mathbb{E}\left(\exp\left(-t\ E_1\left(H_\Lambda^D(\omega)\right)\right)\right)\geq\exp\Bigg(G(t)+t\sup_{1\leq l\leq L}\left(-4d\sin^2\left(\frac{\pi}{2}\,\frac{1}{l+1}\right)+S(l^{-d},t)\right)\Bigg)$$

and the only remaining problem is to maximize

$$\begin{aligned}
&-4dt\sin^2\left(\frac{\pi}{2}\,\frac{1}{l+1}\right)+tS(l^{-d},t)\\
&\qquad\sim-4dt\sin^2\left(\frac{\pi}{2}\,\frac{1}{l+1}\right)+\begin{cases}c\,(l^{-d\rho}-1)t^{\rho+1} & \rho>0\\ -cdt\ \log(l) & \rho=0\\ c\,(1-l^{-d\rho})t^{\rho+1} & -1<\rho<0\\ -c\,l^d\log(t)\end{cases}
\end{aligned}$$

with respect to l. The exponent of the time t in the scaling expression is responsible for the occurrence of the classical or the quantum regime. In the case $\rho>0$, the scaling term increases faster in t than the linear time dependence in the diffusion term. Consequently, the maximum will be asymptotically $l=1$. This corresponds to the classical regime. In the case $-1<\rho<0$ as well as in the fourth situation, the scaling term is sublinear and the diffusion term is dominating. Like in Section 3.2, a collective behavior of potential values is necessary and we are in the quantum regime.

In the case $\rho = 0$, the diffusion and the scaling term are both linear in time. So the optimal peak size depends strongly on the constants. This is the borderline between the classical and the quantum regime. It corresponds to the double exponential distribution.

The upper bounds are much more complicated. It is not possible to interchange the supremum and the expectation with respect to the random potential. Moreover, we have to to estimate (3.38) for all $p \in M_1(\Lambda)$. The first problem is solvable by a variant of the ordinary Laplace method. The second problem is attacked using the convexity of the cumulant generating function $G(t)$ and ideas from spectral geometry. For details, we refer to [**125**] and [**126**].

3.6. Internal Band Edges. Lifshitz predicted the "Lifshitz behavior" not only for the bottom of the spectrum but also for other band edges. We refer to this phenomenon as "internal Lifshitz tails." Internal Lifshitz tails have been proven for the Anderson model by Mezincescu [**128**] and Simon [**157**]. Their proofs apparently cannot be translated to the continuum case. In fact, the band edges of the Anderson model which they can handle are those coming from gaps in $\operatorname{supp} P_0$ together with the boundedness of the kinetic energy. (To be more precise: Since for the Anderson model, $\| h_0\| \leq 4d$ and $h_0 \geq 0$, there are gaps in the spectrum whenever there are gaps in $\operatorname{supp} P_0$ of length exceeding $4d$).

One can also handle the case of a point interaction potential in one dimension, a problem which essentially reduces to a lattice problem. Formally this potential is given by

$$V_\omega = \sum q_i \delta(x - i)$$

where δ is the Dirac-"function." This potential is also known as the random *Kronig–Penney model*. It turns out in this case that the lower edges do and the upper edges (for $q_i \geq 0$) do not show Lifshitz behavior but polynomial behavior of N as for periodic potentials [**79**]. This is due to the fact that the upper edges are "stable boundaries" in the sense of [**148**]. The case of general one-dimensional alloy-type potentials was treated in [**130**].

The multidimensional case is by far more difficult. The reason is mainly that *periodic* potentials are much less well understood in higher dimensions. For example, it is not true in general that bands are parabolic, as is the case for $d = 1$ and for the ground state band in arbitrary dimension.

The paper [**97**] marks a breakthrough in this topic. Klopp uses the method of approximation by periodic potentials. Compared to Dirichlet–Neumann bracketing, one loses monotonicity, a property which was very useful above. However, Klopp manages to prove an exponential convergence rate for the periodic approximations.

As mentioned above, not so much is known about the behavior of the band functions (of the periodic operators) at internal band edges. In fact,

Klopp has to make assumptions on the behavior of the integrated density of states for the *periodic* operator.

Like Lifshitz tails at the bottom of the spectrum internal Lifshitz tails can be used as an input for a localization proof [**175**].

We consider an alloy-type potential with a continuous single site potential $f \geq 0$, not identically equal to 0, with decay

$$f(x) \leq C\,(1 + |\,x|)^{-(d+2+\varepsilon)}. \tag{3.44}$$

The random coupling constants are independent and have a common probability distribution P_0 with $q_{\min} = \inf \operatorname{supp} P_0$. We set $V_{\rm per} = \sum_{i\in\mathbb{Z}^d} q_{\min}\, f(x-i)$ and denote the integrated density of states of $H_{\rm per} = H_0 + V_{\rm per}$ by $N_{\rm per}(E)$. Furthermore, we suppose that E_- is a lower band edge of $H_{\rm per}$, i.e., $E_- \in \sigma(H_{\rm per})$, but $(E_- - a, E_-) \cap \sigma(H_{\rm per}) = \emptyset$.

It is reasonable to assume that generically $N_{\rm per}$ behaves like $(E-E_-)^{d/2}$, for $E \searrow E_-$, as it would for a unique parabolic band. In fact, this behavior is known in one dimension and for the bottom of the spectrum in arbitrary dimension. However, it is not clear that this is true in general, even not generically (see, however, [**103**]).

Thus, we have to *assume* such a behavior of $N_{\rm per}$:

Assumption: Suppose E_- is a lower band edge of $H_{\rm per}$. We assume that

$$\lim_{E\searrow E_-} \frac{\ln\,\big(\,N_{\rm per}(E) - N_{\rm per}(E_-)\,\big)}{\ln\,(\,E - E_-\,)} = -\frac{d}{2}\,. \tag{3.45}$$

Under this assumption, the main result of [**97**] is:

THEOREM 3.8 (Klopp). *If assumption* (3.45) *holds, then*

$$\lim_{E\searrow E_-} \frac{\ln\,\Big(-\ln\,\big(\,N(E) - N(E_-)\,\big)\,\Big)}{\ln\,(\,E - E_-\,)} = -\frac{d}{2}\,. \tag{3.46}$$

For the case $d = 2$, one has more information about the periodic operators [**105**]. In particular, there is always exponential decay of the integrated density of states at band edges. We refer to the review [**100**] for an introduction and further results.

3.7. Lifshitz Tails for Surface Potentials. In this section we consider surface potentials of the form

$$V^s_\omega(x_1, x_2) = \sum_{i_1\in\mathbb{Z}^{d_1}} q_{i_1}(\omega) f(x_1 - i_1, x_2) \tag{3.47}$$

and suppose we have some spectrum below 0. This is the case if $q_{\min} = \inf \operatorname{supp} P_0$ is negative enough. Note that for $d_2 \leq 2$, there is negative spectrum as soon as $q_{\min}$ is negative. For $d_2 \geq 3$, there is a threshold $\gamma > 0$ such that the spectrum starts at 0 if $q_{\min} \geq -\gamma$ and there is negative spectrum if $q_{\min} < -\gamma$.

We are going to investigate Lifshitz tails for surface potentials with $E_0 < 0$. Below the bulk spectrum (which starts at 0), the density of surface states

is positive, hence a measure. We may therefore define the integrated density of surface states $N_s(E)$ to be the corresponding distribution function.

As before, we decompose the potential into a non-random background potential and a positive random potential

$$\begin{aligned} V^s_\omega(x_1,x_2) &= \sum_{i_1\in\mathbb{Z}^{d_1}} q_{i_1}(\omega) f(x_1-i_1,x_2) \\ &= \sum_{i_1\in\mathbb{Z}^{d_1}} q_{\min}\, f(x_1-i_1,x_2) + \sum_{i_1\in\mathbb{Z}^{d_1}} (q_{i_1}-q_{\min})\, f(x_1-i_1,x_2) \\ &= V^s_{\rm sp}(x) \;+\; \tilde V^s_\omega(x). \end{aligned} \tag{3.48}$$

The Neumann–Dirichlet bracketing technique goes through for this case as soon as we have sufficient knowledge about the background operator $H_1 = H_0 + V^s_{\rm sp}$. Since we want $E_0 < 0$ —which makes the bottom of the spectrum "surface spectrum"—there is no case $V^s_{\rm sp} = 0$ here. Moreover, the background potential $V^s_{\rm sp}$ is only periodic for the d_1-directions, but decays perpendicular to them.

The analysis of those partially periodic potentials and the Lifshitz estimates for surface potentials were done in [**87**] for the continuous case. We assume that

$$P_0\big([q_{\min}, q_{\min}+\varepsilon)\,\big) \;\geq C\,\varepsilon^N$$

and

$$0 \;\leq\; f(x_1,x_2) \;\leq\; f_0\,(1+|\,x_1|)^{-(d_1+2)}.$$

We also assume that $f(x_1,x_2)$ decays uniformly in x_2-directions. Then we have:

THEOREM 3.9. *If $E_0 < 0$, then*

$$\lim_{E\searrow E_0} \frac{\ln\big(-\ln(N_s(E))\big)}{\ln(E-E_0)} = -\frac{d_1}{2}\,. \tag{3.49}$$

There is also an analogous theorem for long range f. The paper [**72**] proves Lifshitz tails for surface potentials in the discrete setting by fairly different techniques. This paper also contains an analysis at the energy $E = 0$, i.e., for surface corrections to the bulk Lifshitz tails.

3.8. Lifshitz Tails for Random Landau Hamiltonians. We turn to the density of states for operators of the form

$$H_\omega = H_0(B) + V_\omega$$

with a constant magnetic field $B > 0$ and a non-negative random potential V_ω. (For a careful definition of the density of states and some basics, see [**62, 63, 178**].)

We discuss the *two-dimensional* case first. The Landau Hamiltonian $H_0(B)$ is given by

$$H_0(B) = \left(i\frac{\partial}{\partial x_1} - \frac{1}{2}Bx_2\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{1}{2}Bx_1\right)^2.$$

$H_0(B)$ has a pure point spectrum for $B \neq 0$ and $d = 2$. In fact, the eigenvalues are given by the "Landau levels" $(2n+1)B$; $n \in \mathbb{N}$ and all Landau levels are infinitely degenerate. One possible choice of the ground state is

$$\psi_0(x) = \frac{B}{\pi}\, e^{-\frac{B}{2}|x|^2} \tag{3.50}$$

which will play a major role below. For V_ω we take a Poisson potential or an alloy-type potential with $q_{\min} = 0$. In this case, the bottom E_0 of the spectrum of H_ω is given by the lowest Landau level, which is B. In [**10**] the authors proved for the Poisson model the following result.

THEOREM 3.10. *If $B \neq 0$ and*

$$\frac{C_1}{(1+|x|)^\alpha} \leq f(x) \leq \frac{C_2}{(1+|x|)^\alpha}, \tag{3.51}$$

the Lifshitz exponent η for $H_0(B) + V_\omega$ is given by

$$\eta = \frac{2}{2-\alpha} \quad \left(= \frac{d}{d-\alpha}\right) \tag{3.52}$$

for all $\alpha > d$.

This means that, according to our classification above, we are always in the classical case for $d = 2$ and constant magnetic field.

PROOF. We sketch the lower bound only and restrict ourselves to the alloy-type case. As usual we have to estimate

$$\mathbb{P}(E_1(H_{\omega\,\Lambda_L}^D) < E_0 + E)$$

from below. This time we have a ground state ψ_0 for $H_0(B)$ which is L^2. We modify ψ_0 near the boundary of Λ_L to make it satisfy Dirichlet boundary conditions. Due to the (super-)exponential decay of ψ_0, the error we make is of the order (at most) $e^{-C_0\, L^2}$. Thus

$$\begin{aligned}
\mathbb{P}(E_1(H_{\omega\,\Lambda_L}^D) < E_0 + E) &\geq \mathbb{P}\left(\int V_\omega(x)|\psi_0|^2 dx < E - e^{-C_0\, L^2}\right) \\
&\geq \mathbb{P}\left(\int V_\omega(x) dx < C_1 E - e^{-C_2\, L^2}\right) \\
&= (*)\ .
\end{aligned}$$

At this point we can literally repeat the estimates in the proof of Theorem 3.5 and obtain in analogy to (3.25)

$$(*) \geq \mathbb{P}\Big(\frac{1}{|\Lambda_{2L}|} \sum_{|i|\leq 2L} q_i \leq C_1 E - C_2 e^{-cL^2} - C_3 L^{-(\alpha-d)}\Big)\ . \tag{3.53}$$

The only difference to the previous case is the error term due to the kinetic energy. In Theorem 3.5 it was of the order L^{-2} causing the different behavior for $\alpha \geq d+2$ and $\alpha < d$. In (3.53) the error term is exponentially small, thus being negligible with respect to the potential term $L^{-(\alpha-d)}$ for *all* α.

Consequently, we may choose $E \sim L^{\frac{1}{\alpha-d}}$. By a large deviation estimate, we obtain

$$\begin{aligned}(3.53) &\geq \mathbb{P}(q_i \leq C'E)^{|\Lambda_{2L}|} \\ &\geq Me^{-\tilde{C}L^d} \\ &= Me^{-\tilde{\tilde{C}}E^{-\frac{d}{\alpha-d}}}. \end{aligned}$$ □

Theorem 3.10 implies that for compactly supported f, the integrated density of states $N(E)$ decays subexponentially.

Erdös [**38, 39**] proved that it decays, in fact, polynomially. Erdös' proof is based on a careful estimate of the Laplace transform of N. It uses an analog of the Feynman–Kac formula for magnetic Schrödinger operators, the Feynman–Kac–Ito formula (see [**154**]). There are a couple of complications in the Feynman–Kac expression of $\tilde{N}$ due to the magnetic field. The most serious one is the fact that the integrand is no longer positive but rather oscillating.

Erdös' proof is done for the Poisson model. Recently, Klopp and Raikov [**102**] found a completely different approach based on the approximation by periodic potentials. Their proof works for alloy-type potentials. Moreover, the latter paper contains results for internal Lifshitz tails as well as for the case of an additional periodic background potential.

We state Erdös' result:

THEOREM 3.11. *If $H_\omega = H_0(B) + V_\omega$ with a Poisson random potential V_ω and if the single site potential $f \geq 0$ has compact support, then*

$$\lim_{E \searrow E_0} \frac{\ln N(E)}{\ln E} = -\frac{\pi}{B}. \tag{3.54}$$

The reasoning we gave in the proof of Theorem 3.10 can be used to prove a lower bound for Theorem 3.11 as well. In fact, the bound suggests that the borderline between the two kinds of behavior is given by Gaussian single site potentials. This was actually proved in [**64**] and [**39**].

We have seen in this section that random Landau Hamiltonians may have unusual Lifshitz behavior compared to the nonmagnetic case. This originates in the fact that the lowest Landau band is flat, collapsing into one point. It is well known that this never happens for ($B = 0$ and) periodic *scalar* potential (see [**150**]).

It is reasonable to expect that the flatness of the ground state Landau band is removed by an additional periodic scalar potential, at least generically. If this is true, we would certainly expect "normal" Lifshitz behavior for such operators, including those with alloy-type potentials and $q_{\min} \neq 0$. There are results in this direction in [**102**].

We turn to the *three-dimensional* case with homogeneous magnetic field. We consider the Hamiltonian

$$H_0(B) = \left(i\frac{\partial}{\partial x_1} - \frac{1}{2}Bx_2\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{1}{2}Bx_1\right)^2 - \frac{\partial^2}{\partial {x_3}^2}\,.$$

In $d = 3$, the magnetic field itself introduces an anisotropy. The two space dimensions perpendicular to the magnetic field (x_1, x_2) will be denoted by $x_\perp$, the direction x_3 parallel to the field by $x_\parallel$. If we add an anisotropic Anderson (or Poisson) potential with a single site potential f obeying

$$\frac{a}{|x_\perp|^{\alpha_1} + |x_\parallel|^{\alpha_2}} \leq f(x) \leq \frac{b}{|x_\perp|^{\alpha_1} + |x_\parallel|^{\alpha_2}}\,, \tag{3.55}$$

we will have Lifshitz behavior that resembles the results of Section 3.4, except that in the $\perp$-direction the behavior is always *classical*.

Following the conventions of Section 3.4, we define

$$\gamma_1 = \frac{2}{\alpha_1}\,,\ \ \gamma_2 = \frac{1}{\alpha_2}\ \text{ and }\ \gamma = \gamma_1 + \gamma_2. \tag{3.56}$$

If f has compact support in $x_\perp$-direction, we set $\gamma_1 = 0$. To have the potential well defined, we need $\gamma < 1$.

THEOREM 3.12. *The Lifshitz exponent η for $H_0(B) + V_\omega$ in dimension $d = 3$ is given by*

$$\eta = \frac{\gamma_1}{1-\gamma} + \max\left\{\frac{1}{2}\,,\ \frac{\gamma_2}{1-\gamma}\right\}. \tag{3.57}$$

This theorem was proven for f with compact support in [**178**]. The case of $\gamma_2 > \frac{1}{2}$ is considered in [**60**] and the mixed classical-quantum case is taken from [**73**].

We remark that the lower bound given above (in connection with Theorem 3.10) can be used in this case as well. As a test function in this bound, we use

$$\Psi_0(x_\perp, x_\parallel) = \psi_0(x_\perp)\,\phi_0(x_\parallel) \tag{3.58}$$

where ψ_0 is the (2-d) ground state (3.50) and $\phi_0(x_\parallel) = L^{-1/2}$ is the $\parallel$-ground state. As before Ψ_0 has to be cut down to zero near the boundary of the cube Λ_L. This gives an error term of the order e^{-L^2} in the $\perp$-direction and an error term of the order L^{-2} in the $\parallel$-direction.

For further references about magnetic Lifshitz tails, we refer to [**178**] and to [**116**].

There are also results on the integrated density of states and on Lifshitz tails for *random* magnetic fields. We refer to the works of Ueki and Nakamura [**172, 173, 174, 138, 139**] and the references given there.

3.9. Percolation Models. Recently the integrated density of states for Laplacians on percolation graphs has been investigated. We consider bond percolation on $\mathbb{Z}^d$ with $d \geq 2$. This means that we remove bonds in the graph $\mathbb{Z}^d$ independently with probability $1-p$ for $0 < p < 1$. The resulting random graph is called the *percolation graph* and is denoted by $\mathcal{G}_p$ or simply by $\mathcal{G}$.

For p small ($p < p_c$), all connected components ("clusters") of $\mathcal{G}_p$ are finite almost surely. For $p > p_c$ there is a unique infinite cluster (see, e.g., [**54**]).

We consider the Laplacian on this graph which is a random operator, due to the randomness of the underlying graph. For a general discussion of percolation Hamiltonians, we refer to [**176**].

Actually, there are various Laplacians on $\mathcal{G}$ due to different boundary conditions. To define these operators, we start with the adjacency operator $A_\mathcal{G}$ defined by the matrix elements with $A_\mathcal{G}(i,j) = 1$ if $|i-j| = 1$, $i, j \in \mathcal{G}$ and $A_\mathcal{G} = 0$ otherwise. For any $i \in \mathcal{G}$, we let $d_\mathcal{G}(i)$ denote the number of sites in $\mathcal{G}$ to which i is connected. We denote by $D_\mathcal{G}$ the diagonal matrix with entries $d_\mathcal{G}(i)$ on the diagonal. The Neumann Laplacian $L^N_\mathcal{G}$ is defined by

$$L^N_\mathcal{G} = D_\mathcal{G} - A_\mathcal{G}. \tag{3.59}$$

The Neumann Laplacian is the intrinsic Laplacian of the graph $\mathcal{G}$. In a sense it "ignores" the embedding of $\mathcal{G}$ into $\mathbb{Z}^d$. The Dirichlet Laplacian $L^D_\mathcal{G}$ is defined by

$$L^D_\mathcal{G} = D_\mathcal{G} + 2\,(2d - D_\mathcal{G}) - A_\mathcal{G}. \tag{3.60}$$

On its diagonal this operator counts the connections to neighboring sites *in* $\mathcal{G}$ once and the (lost) connections to sites in $\mathbb{Z}^d \setminus \mathcal{G}$ twice. It was an observation of Simon [**156**] that the above operator is a good analog of the Dirichlet Laplacian for subgraphs of $\mathbb{Z}^d$. See also [**78**] or [**70**] for a discussion of boundary conditions for discrete Laplacians.

Both for $L^N_\mathcal{G}$ and for $L^D_\mathcal{G}$, a integrated density of states can be defined in analogy to Theorem 2.2. We call them $N^N_\mathcal{G}(E)$ and $N^D_\mathcal{G}(E)$, respectively.

Since there are infinitely many clusters containing just one point, the Neumann Laplacian $L^N_\mathcal{G}$ has an eigenvalue of infinite multiplicity at the bottom $E = 0$ of its spectrum. This causes the integrated density of states to jump at that energy. Hence

$$\nu^N_\mathcal{G}(\{0\}) = N^N_\mathcal{G}(0) - N^N_\mathcal{G}(0-) = N^N_\mathcal{G}(0) \ \neq 0.$$

THEOREM 3.13. (1) *If $p < p_c$, we have*

$$\lim_{E \searrow 0} \frac{\ln\Big(-\ln\big(N^N_\mathcal{G}(E) - N^N_\mathcal{G}(0)\big)\Big)}{\ln(E)} = -\frac{1}{2}. \tag{3.61}$$

(2) *If $p > p_c$, we have*

$$\lim_{E \searrow 0} \frac{\ln\big(N^N_\mathcal{G}(E) - N^N_\mathcal{G}(0)\big)}{\ln(E)} = \frac{d}{2}. \tag{3.62}$$

(3) *For arbitrary* $0 < p < 1$, *we have*

$$\lim_{E \searrow 0} \frac{\ln\Big(-\ln N_{\mathcal{G}}^{D}(E)\Big)}{\ln(E)} = -\frac{d}{2}. \tag{3.63}$$

Part (1) of this theorem, as well as part (3) for $p < p_c$, was proven in [**78**]; part (2) and part (3) are taken from [**133**].

The perhaps surprising behavior of $N_{\mathcal{G}}^{N}(E)$ for $p < p_c$ is due to the fact that long one-dimensional chains dominate. They lead to small eigenvalues for the Neumann Laplacian since they "don't know" they are actually in $\mathbb{Z}^d$. For the Dirichlet Laplacians these one-dimensional chains have rather high eigenvalues due to the additional diagonal term. In fact, they dominate the scene at $E = 4d$, the top of the spectrum. This can be seen by a symmetry argument.

Part (2) of the theorem comes from the fact that for $p > p_c$ the leading behavior of $N_{\mathcal{G}}^{N}$ comes from the infinite cluster. In a sense, the infinite cluster looks rather d-dimensional. The proof in [**133**] of this part relies on a celebrated paper by Barlow [**5**].

4. Regularity of the Integrated Density of States

4.1. Introduction. In this final section we discuss regularity properties of the integrated density of states. So far we have seen that the density of states measure ν is a positive Borel measure with a distribution function $N(E)$.

Of course, the name integrated *density* of states suggests that ν (resp. N) should have a density $n(E)$ in the sense

$$N(E) = \int_{E' \leq E} n(E')\, dE', \tag{4.1}$$

$$\nu([a,b]) = \int_{[a,b]} n(E')\, dE'. \tag{4.2}$$

We will, indeed, prove this for the *Anderson model* under certain assumptions on P_0, the probability distribution of the random potential. The continuous case is more complicated. For this case we will state only some key results and refer the reader to the literature. A good account is the review article by Veselic [**177**]. However, we remark that there are important developments after this survey; we mention especially the recent preprint [**20**].

We will see that the integrated density of states does *not* always have a density. We will therefore also look at weaker regularity properties of N. We are especially interested in the question whether the function N is continuous, which is the same as $\nu(\{E\}) = 0$ for all E. It turns out that this is always the case for the Anderson model. However, for the continuous case, no such result is known.

4.2. Continuity of the Integrated Density of States. The results of this section are valid for general ergodic operators on $\ell^2(\mathbb{Z}^d)$. Analogous results for $\mathbb{R}^d$ have not been proven so far. The most general result in this context is the following theorem proved by Craig and Simon [**25**] in a somewhat stronger form (see Theorem 4.3).

THEOREM 4.1 ([**25, 30**]). *Let $\{V_\omega(n)\colon n \in \mathbb{Z}^d\}$ be an ergodic stationary real valued random potential satisfying*

$$\mathbb{E}[\log(1 + |V_\omega(0)|] < \infty. \tag{4.3}$$

Then the integrated density of states of the Anderson operator $h_\omega = h_0 + V_\omega$ is a continuous function, i.e., $\nu(\{\lambda\}) = 0$ for all $\lambda \in \mathbb{R}$.

This result can be proven with an elementary argument by Delyon and Souillard [**30, 27**] using a kind of "unique continuation" property of discrete Schrödinger operators. We use this idea in the proof of the following lemma to show that no eigenspace can be sufficiently degenerated to produce a jump of the integrated density of states.

Let us denote by $\mu_H(\cdot)$ the projection-valued spectral measure of H, i.e., $\mu_H(A) = \chi_A(H)$; in particular, $\mu_H(\{\lambda\})$ is the projector onto the eigenspace of H with respect to λ.

LEMMA 4.2. *We have*

$$\dim\big(\chi_{\Lambda_L}\mathrm{Ran}\,\mu_{h_\omega}(\{\lambda\})\big) \le C\,L^{d-1}.$$

PROOF. The set

$$\Lambda_L^{(2)} = \left\{ i \in \Lambda_L \big| \quad \max_{\nu=1,\dots,d} |i_\nu| = L \ \text{ or } \max_{\nu=1,\dots,d} |i_\nu| = L-1 \right\}$$

consists of the two outermost layers of Λ_L. A solution u of $h_\omega u = \lambda u$ is uniquely determined inside Λ_L by its values on $\Lambda_L^{(2)}$. So, the dimension of $\chi_{\Lambda_L}\big(\mathrm{Ran}\mu_{h_\omega}(\{\lambda\})\big)$ is at most the number of points in $\Lambda_L^{(2)}$. □

With the lemma, we can prove the theorem.

PROOF OF THE THEOREM. By Proposition 2.1 and Theorem 2.2, we have

$$\nu(\{\lambda\}) = \lim_{L\to\infty} \frac{1}{(2L+1)^d}\,\mathrm{tr}\,\big(\chi_{\Lambda_L}\,\mu_{h_\omega}(\{\lambda\})\big). \tag{4.4}$$

If f_i is an orthonormal basis of $\chi_{\Lambda_L}\big(\mathrm{Ran}\,\mu_{h_\omega}(\{\lambda\})\big)$ and g_i an orthonormal basis of $\big(\chi_{\Lambda_L}\big(\mathrm{Ran}\,\mu_{h_\omega}(\{\lambda\})\big)\big)^\perp$ we have, noting that $\chi_{\lambda_L}(\mathrm{Ran}\,\mu_{h_\omega}(\{\lambda\}))$ is finite-dimensional,

$$\begin{aligned}
\mathrm{tr}\,\big(\chi_{\Lambda_L}\,\mu_{h_\omega}(\{\lambda\})\big) &= \sum_{i\in I}\langle f_i, \chi_{\Lambda_L}\,\mu_{h_\omega}(\{\lambda\})f_i\rangle \\
&\le \dim\big(\chi_{\Lambda_L}\big(\mathrm{Ran}\,\mu_{h_\omega}(\{\lambda\})\big)\big) \qquad (4.5)\\
&\le C\,L^{d-1} \qquad (4.6)
\end{aligned}$$

hence (4.4) converges to zero and $\nu(\{\lambda\}) = 0$. □

In [**25**] Craig and Simon prove a stronger result than just continuity:

THEOREM 4.3 (Craig–Simon). *Under the assumptions of Theorem 4.1, the integrated density of states is locally log-Hölder continuous, e.g., for any positive r there exists a finite constant C_r such that*

$$|N(\lambda) - N(\lambda')| \leq C_r \, \frac{1}{|\log|\lambda - \lambda'|\,|} \tag{4.7}$$

for $|\lambda| \leq r$ and $|\lambda - \lambda'| \leq 1$.

The theorem of Craig and Simon is based on the Thouless formula for a strip in $\mathbb{Z}^d$. For the Thouless formula in the one-dimensional setting, we refer to the next subsection.

4.3. Regularity of the DOS in Dimension One. There are very powerful techniques to study the Schrödinger equation and its discrete analog in dimension *one* which are unfortunately restricted to dimension one exclusively. Such techniques have been successfully applied to random operators as well to investigate the integrated density of states.

At the heart of most of these techniques lies the reformulation of the eigenvalue equation of a second-order equation into an initial value problem for a system of first-order equations. Specifically, let us look at the eigenvalue equation

$$h_\omega = -u(n+1) - u(n-1) + (V_\omega(n) - E)u(n) = 0. \tag{4.8}$$

We define

$$U(n) = \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix}$$

and

$$A_n(E) = \begin{pmatrix} V_\omega(n) - E & -1 \\ 1 & 0 \end{pmatrix}.$$

The function $u(n)$ is a solution of (4.8) if and only if

$$U(n+1) = A_{n+1}(E)U(n)$$

for all $n \in \mathbb{Z}$. With the transfer matrix

$$\Phi_n(E) := \prod_{i=1}^{n} A_i(E)$$

the solution of (4.8) to the right initial condition

$$U(0) = \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$$

can be expressed by

$$U(n) = \Phi_n(E)U(0).$$

Similarly, it is possible to define the solution to the left. The spectral theory of the operator h_ω is encoded in the matrices $A_n(E)$ (or $\Phi_n(E)$). Note that these matrices belong to the group $SL(2, \mathbb{R})$. Thus it should not come as

a surprise that harmonic analysis on $SL(2,\mathbb{R})$, in explicit or implicit form, plays a major role in the analysis of h_ω.

The asymptotic behavior of the eigensolutions of $h = h_0 + V$ is reflected by the Lyapunov exponent given by

$$\gamma(E) := \lim_{N\to\pm\infty} \ln \|\Phi_n(E)\|. \tag{4.9}$$

This definition of the Lyapunov exponent is well defined by Fürstenberg's theorem ([**46**], see also [**27**]). The link between the Lyapunov exponent and the integrated density of states is expressed in the Thouless formula

$$\gamma(E) = \int \log|E - E'| \, dN(E'). \tag{4.10}$$

This formula from physics ([**56, 171**]) was made rigorous in [**3**]. A simplified proof can be found in [**26**] or [**27**].

The Thouless formula and the obvious fact that $\gamma(E)$ is non-negative imply that the integrated density of states N is log-Hölder continuous (in one dimension). This was observed by Craig and Simon in [**26**]. The proof by the same authors of the multidimensional analog uses a version of the Thouless formula for strips in higher dimensions [**25**].

As mentioned in the previous section, these results hold for general ergodic potentials on $\mathbb{Z}^d$, they are (at least) close to optimal in this generality (see also our discussion below).

However, if we assume that the $V_\omega(n)$ are independent (and identically distributed) much more can be said about the regularity of the integrated density of states. In the next section, we discuss Lifshitz continuity in the multidimensional case under assumptions on the distribution P_0 of the random variable $V_\omega(0)$. In the one-dimensional case we discuss here, we have a fairly complete picture about the regularity of N. For simplicity we assume that $\operatorname{supp} P_0$ is compact.

Definition 4.4. *We call a function f on $\mathbb{R}$ Hölder continuous of order α if*

$$|f(x) - f(y)| \le C\,|x - y|^\alpha.$$

Theorem 4.5. *Let $h = h_0 + V$ be a discrete random Schrödinger operator in dimension one. Assume V is a sequence of independent, identically distributed random variables, such that their distribution P_0 has compact support. Then the integrated density of states N of h is Hölder continuous of some order $\alpha > 0$.*

This theorem is due to [**114**]. The simple case of the discrete Laplacian h_0 (i.e., $V = 0$) shows that one cannot expect more than Hölder continuity in general. For h_0 the integrated density of states N is merely Hölder continuous of order $\frac{1}{2}$ at the band edges. The derivative of N, which is called the *density of states*, diverges at $E = \pm 2$. This behavior is known as van Hove singularities. The Lifshitz behavior of the integrated density of states

seems to suggest that N is smoother at the band edges for a truly random V.

One can expect that more regularity for P_0 implies more regularity for N. In fact, Simon and Taylor [**161**] proved that already a little regularity of P_0 implies that N is C^∞. To state their result, we need the following definition of a Sobolev space:

DEFINITION 4.6. *We say that a function $f \in L^p$ belongs to L^p_α if the Fourier transform $\hat{f}$ satisfies: There is a function $g \in L^p$ such that $\hat{g}(k) = (1+|k|^2)^{\alpha/2}\hat{f}(k)$ is an L^p-function.*

It is not hard to see that the characteristic function χ_I of a finite interval I is in L^1_α for $\alpha < \frac{1}{2}$ (see [**161**]).

THEOREM 4.7. *Let $h = h_0 + V$ be a discrete random Schrödinger operator in dimension one. Assume V is a sequence of independent, identically distributed random variables. If the distribution P_0 of $V_\omega(0)$ has a density g with compact support and such that $g \in L^1_\alpha$ for some $\alpha > 0$, then $N \in C^\infty$.*

If we do not assume any regularity of P_0, Theorem 4.5 is the best result we can hope for. To see this, let us look at a Bernoulli distribution for P_0. We set

$$P_0 = p\,\delta_a + (1-p)\delta_b. \tag{4.11}$$

Halperin [**55**] proved, with some points of rigor clarified by Simon and Taylor [**161**], that:

THEOREM 4.8. *Assume that the $V_\omega(n)$ are independent with identical distributions P_0 as in* (4.11) *with $0 < p \le \frac{1}{2}$. Then the integrated density of states is* not *Hölder continuous of any order α larger than*

$$\alpha_0 = \frac{2\,|\log(1-p)|}{\operatorname{arccosh}(1+\frac{1}{2}|b-a|)}.$$

If N is Lifshitz continuous (Hölder continuous of order $\alpha = 1$), then N has a bounded density n, i.e., $N(E) = \int_{-\infty}^{E} n(\lambda)\,d\lambda$ and vice versa. If N is Hölder continuous of (strict) order $\alpha < 1$, then N may still have a density which then has to be unbounded. So, Theorem 4.8 does not rule out that the density of states measure ν is absolutely continuous. However, it is true that ν has a singular continuous component if $|b-a|$ is large.

THEOREM 4.9. *If P_0 is Bernoulli (as in* (4.11)*) and $0 < p < 1$, then the density of states measure ν has a singular continuous component if $|b-a|$ is large.*

This theorem was proved in [**14**] following ideas from [**161**]. The paper [**14**] contains the first proof of Anderson localization for the one-dimensional Bernoulli model. The proof of a singular continuous component of N is based on this knowledge. The article [**122**] continues these investigations

and proves that for $|b-a|$ large, the density of states measure is even purely singular continuous.

For further results concerning the regularity of the DOS of random Schrödinger operators and its discrete analog in dimension one see [**12, 28, 13, 88, 90, 91, 92, 93, 94, 120, 153**].

As the operators with i.i.d. potentials, *almost periodic* operators belong to the class of ergodic operators. In a sense these two types of operators form the extreme cases within the class of ergodic operators. There are recent results on regularity of the integrated density of states for one-dimensional almost periodic operators by Goldstein and Schlag [**51, 53**]. We refer to the review [**52**] for details.

4.4. The Wegner Estimates: Discrete Case. One motivation in physics to study the integrated density of states was the hope to use it as an indicator for different spectral types. The aim was to distinguish pure point spectrum and continuous spectrum at the mobility edge. In some sense the conjectures discussed were inconsistent. Some expected a divergent density of states at the mobility edge, while others assumed a vanishing density of states.

In 1981 Wegner [**179**] put an end to this discussion by proving upper and lower bounds of the density of states in the discrete setting of the Anderson model. These estimates imply the density of states neither vanishes nor explodes at a mobility edge (or anywhere else).

Wegner's result (more precisely, his upper bound) soon became a corner piece in the proofs of Anderson localization by the multiscale analysis method and it still is. To formulate Wegner's result, we introduce boundary conditions on the lattice. The simplest boundary condition on the lattice is defining h_ω^Λ through its matrix elements:

$$h_\omega^\Lambda(i,j) = h_\omega(i,j) \tag{4.12}$$

whenever both i and j belong to Λ. For a discussion of boundary condition on ℓ^2, see Simon [**156**] or the review [**70**].

We define

$$N_\Lambda(E) = \#\{\, n \,|\, E_n(h_\omega^\Lambda) \le E\} = \operatorname{tr}\left(P_{(-\infty,E]}(h_\omega^\Lambda)\right). \tag{4.13}$$

THEOREM 4.10 (Wegner-estimate). *Suppose the measure P_0 has a bounded density g, (i.e., $P_0(A) = \int_A g(\lambda)d\lambda$, $||g||_\infty < \infty$), then*

$$\mathbb{E}(N_\Lambda(E+\varepsilon) - N_\Lambda(E-\varepsilon)) \le 2\,||g||_\infty\,|\Lambda|\,\varepsilon. \tag{4.14}$$

REMARK 4.11. *The assumption that P_0 has a density cannot be dropped. We have seen at the end of Section 4.3 that the integrated density of states has a singular continuous component if P_0 is a (certain) Bernoulli distribution.*

Before we discuss this estimate, we note two important consequences. The first concerns the regularity of the density of states.

COROLLARY 4.12. *Under the assumption of Theorem 4.10, the integrated density of states is absolutely continuous with a bounded density $n(E)$.*

PROOF.

$$\begin{aligned} N(E+\varepsilon) - N(E-\varepsilon) &= \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|}\mathbb{E}(N_\Lambda(E+\varepsilon) - N_\Lambda(E-\varepsilon)) \\ &\le C\varepsilon \qquad \text{by Theorem 4.10.} \end{aligned}$$ □

Thus $N(E) = \int_{-\infty}^{E} n(\lambda)\, d\lambda$. We call $n(\lambda)$ the *density of states*. Sometimes, N also is called the density of states which, we admit, is an abuse of language. The second consequence of Theorem 4.10 is a key ingredient in proving Anderson localization.

COROLLARY 4.13. *Under the assumptions of Theorem 4.10, we have for any E and Λ,*

$$\mathbb{P}(\operatorname{dist}(E, \sigma(h_\omega^\Lambda)) < \varepsilon) \le C\varepsilon|\Lambda|. \tag{4.15}$$

PROOF. By the Chebyshev inequality, we get

$$\begin{aligned} \mathbb{P}(\operatorname{dist}(E, \sigma(h_\omega^\Lambda)) < \varepsilon) &= \mathbb{P}(N_\Lambda(E+\varepsilon) - N_\Lambda(E-\varepsilon) > 1) \\ &\le \mathbb{E}(N_\Lambda(E+\varepsilon) - N_\Lambda(E-\varepsilon)) \\ &\le C\varepsilon|\Lambda| \qquad \text{by Theorem 4.10.} \end{aligned}$$ □

The first step in the proof of Theorem 4.10 is to average the eigenvalues inside the interval $(E-\epsilon, E+\epsilon)$ with respect to the random potential. To do this we consider the eigenvalues $E_n(h_\omega^\Lambda)$ as functions of the arguments $V_i = V_\omega(i)$ with $i \in \Lambda$, i.e., $E_n(h_\omega^\Lambda) = E_n(V_i, i \in \Lambda)$. The resulting estimate is summarized in the following lemma.

LEMMA 4.14. *With $\varepsilon > 0$ let ϱ be a non-decreasing C^∞-function with $\varrho(\lambda) = 1$ for $\lambda > \varepsilon$, $\varrho(\lambda) = 0$ for $\lambda < -\varepsilon$ and $0 \le \varrho(\lambda) \le 1$. Then*

$$N(h_\omega^\Lambda, E+\varepsilon) - N(h_\omega^\Lambda, E-\varepsilon) \le \sum_{j\in\Lambda} \int_{E-2\varepsilon}^{E+2\varepsilon} \frac{\partial}{\partial V_j}\, \operatorname{tr}\varrho(h_\omega^\Lambda(\{V_i\}) - \eta) d\eta.$$

Now we are in a position to interchange the expectation of the random potential and the energy integral. Since the random variables $V_\omega(i)$ are independent and have the common distribution $dP_0(V_i)$, the expectation $\mathbb{E}$

is just integration with respect to the product of these distributions. Hence

$$
\begin{aligned}
&\mathbb{E}(N_\Lambda(E+\varepsilon) - N_\Lambda(E-\varepsilon)) \\
&\quad \leq \mathbb{E}(\sum_{j\in\Lambda} \int_{E-2\varepsilon}^{E+2\varepsilon} \frac{\partial}{\partial V_j} \mathrm{tr}\ (\varrho(h_\omega^\Lambda(\{V_i\}) - \eta) d\eta)) \\
&\quad = \int_{E-2\varepsilon}^{E+2\varepsilon} d\eta \sum_{j\in\Lambda} \prod_{i\in\Lambda} \int_{\mathbb{R}} g(V_i) dV_i \frac{\partial}{\partial V_j}\, \mathrm{tr} \varrho(h_\omega^\Lambda(\{V_i\}) - \eta) \\
&\quad \leq ||g||_\infty \int_{E-2\varepsilon}^{E+2\varepsilon} d\eta \sum_{j\in\Lambda} \prod_{i\in\Lambda,\ i\neq j} \int_{\mathbb{R}} g(V_i) dV_i \quad \int_a^b dV_j \frac{\partial}{\partial V_j} \mathrm{tr} \varrho(h_\omega^\Lambda(\{V_i\}) - \eta) \\
&\quad \leq \int_{E-2\varepsilon}^{E+2\varepsilon} d\eta \sum_{j\in\Lambda} \prod_{i\in\Lambda,\ i\neq j} \int g(V_i)\ dV_i \\
&\qquad\qquad \left\{ \mathrm{tr}\ \varrho(h_\omega^\Lambda(\{V_i\}^{V_j=b}) - \eta) - \mathrm{tr}\ \varrho(h_\omega^\Lambda(\{V_i\}^{V_j=a}) - \eta) \right\}.
\end{aligned}
$$

Above, a, b are such that supp$g \subset [a,b]$. The notation $\{V_i\}^{V_j=b}$ means the family $\{V_i\}_{i\in\Lambda}$ with $\tilde{V}_i = V_i$ for $i \neq j$ and $\tilde{V}_j = b$. The problem is now to estimate the trace difference. In the discrete context of the Anderson model, the variation of a potential value at one site is a rank one perturbation.

LEMMA 4.15. *Let A be a self-adjoint operator bounded below with purely discrete spectrum $E_0 \leq E_1 \leq \ldots$ (where the eigenvalues are repeated according to multiplicity). If B is a symmetric positive rank one operator, then $\tilde{A} = A + B$ has eigenvalue $\tilde{E}_n$ with $E_n \leq \tilde{E}_n \leq E_{n+1}$.*

Given the lemma, we now continue the proof of the theorem. We set $A = H_\Lambda(\{V_i\}^{V_j=a})$ and $\tilde{A} = H_\Lambda(\{V_i\}^{V_j=b})$. Obviously, their difference is a (positive) rank one operator

$$
\begin{aligned}
\mathrm{tr}\ \varrho(\tilde{A} - \eta) - \mathrm{tr}\ \varrho(A - \eta) &= \sum (\varrho(\tilde{E}_n - \eta) - \varrho(E_n - \eta)) \\
&\leq \sum (\varrho(E_{n+1} - \eta) - \varrho(E_n - \eta)) \\
&\leq \sup_{\lambda,\mu}\ \varrho(\lambda) - \varrho(\mu) \\
&= 1. \qquad (4.16)
\end{aligned}
$$

We conclude the proof of Wegner's estimate by proving Lemma 4.15:

PROOF. $B = c\,|h><h|$ with $c \geq 0$, i.e., $B\ \varphi = c\,\langle h, \varphi\rangle\ h$ for some h. By the min-max principle (see [**150**]),

$$\begin{aligned}\tilde{E}_n &= \sup_{\psi_1,\ldots,\psi_{n-1}} \inf_{\substack{\varphi\perp\psi_1,\ldots,\psi_{n-1}\\ ||\varphi||=1}} \langle\varphi, A\varphi\rangle + c\,|\langle\varphi, h\rangle|^2\\ &\leq \sup_{\psi_1,\ldots,\psi_{n-1}} \inf_{\substack{\varphi\perp\psi_1,\ldots,\psi_{n-1},h\\ ||\varphi||=1}} \langle\varphi, A\varphi\rangle\\ &\leq \sup_{\psi_1,\ldots,\psi_{n-1},\psi_n} \inf_{\substack{\varphi\perp\psi_1,\ldots,\psi_{n-1},\psi_n\\ ||\varphi||=1}} \langle\varphi, A\varphi\rangle\\ &= E_{n+1}. \end{aligned} \tag{4.17}$$

□

Wegner's estimate is intimately connected with a method called "spectral averaging." Roughly speaking, spectral averaging says that taking expectation with respect to random parameters will make the spectral measure absolutely continuous. Here is a typical example which comes from the theory of rank one perturbations, see Simon [**158**] and references given there.

To formulate this result, we take a bounded operator h on $\ell^2(\mathbb{Z}^d)$ and set $h_\alpha = h + \alpha\,\delta_j$ for any fixed $j \in \mathbb{Z}^d$. Note that the multiplication operator δ_j is a rank one perturbation of h. We denote by m_α the (projection valued) spectral measure of h_α and set $\mu_\alpha(A) = \langle\delta_j, m_\alpha(A)\ \delta_j\rangle$. We obtain

THEOREM 4.16 (Spectral averaging).

$$\int d\,\mu_\alpha(E)\ d\alpha = dE, \tag{4.18}$$

i.e.,

$$\int\left(\int f(E)\,d\,\mu_\alpha(E)\right)\,d\alpha = \int\ f(E)\,dE$$

for all integrable f.

Wegner's estimate follows from the spectral averaging result (see [**158**]).

Spectral averaging was introduced in the theory of random operators by Kotani [**109**] who used it to prove Anderson localization. He used random boundary conditions in dimension one, but soon Kotani's trick was also used to prove localization in higher-dimensional systems. However, Kotani was not the first to prove a spectral averaging formula. Such a formula was known in the Russian literature earlier, e.g., to Javrjan [**65**] who proved it for "random" boundary conditions. Spectral averaging also plays a prominent role for continuous Schrödinger operators; see [**21**] and references given there.

4.5. Regularity in the Continuous Case. To prove a Wegner estimate for the continuous case is considerably harder than for the discrete case. In fact, only the alloy-type model and a few other cases can be treated

so far (for these cases see [**21**] and [**41**]). For the alloy-type model one can carry over Wegner's original proof ([**69**]).

However, the finite rank estimate (Lemma 4.15) cannot be transferred directly to the continuum (see [**67**]). Thus a direct analog of Wegner's approach only gives

$$\mathbb{E}(N_\Lambda(E+\epsilon) - N_\Lambda(E-\epsilon)) \leq C|\Lambda|^2\epsilon \ . \tag{4.19}$$

The estimate (4.19) obviously gives *no* information on the regularity of N. However, it suffices as input to multiscale analysis to prove Anderson localization (this was observed in [**121**]).

The first to prove a Wegner bound with the "correct" volume dependence were Kotani and Simon in [**110**]. They required the single site potential f to be the characteristic function of the unit cube. Later Combes and Hislop [**18**] relaxed this condition.

Meanwhile, there is a large number of results on "generalized" Wegner estimates of the form

$$\mathbb{E}(N_\Lambda(E+\epsilon) - N_\Lambda(E-\epsilon)) \leq C|\Lambda|^k\epsilon^\alpha \ . \tag{4.20}$$

For $k = 1$, they imply Hölder continuity of the integrated density of states with Hölder exponent α.

We refer to the excellent survey by Veselic [**177**] on the subject which describes the development until 2004. In addition, we mention the papers [**4, 19, 23, 22, 24, 57, 59, 61, 82, 106, 107, 108**] for further reading.

Very recently, Combes, Hislop and Klopp [**20**] published a result which includes (and improves) virtually all previous results.

THEOREM 4.17. *Let q_i be independent random variables with a common distribution P_0 of compact support. Let f be a non-negative single site potential of compact support. Then*

1. *If P_0 is Hölder continuous with Hölder exponent α, then N is Hölder continuous with the same exponent.*
2. *If P_0 is Lifshitz continuous, then N is Lifshitz continuous as well. In this case N has a bounded density.*

The authors of [**20**] actually prove a more general theorem allowing the random variables to be dependent and including a magnetic field.

4.6. Beyond the Density of States: Level Statistics. One may look at the energy statistics of a disordered system on a smaller scale than we do for the integrated density of states. This subject is common in the theory of random matrices since the days of Wigner and Dyson (see, e.g., [**124**] or [**29**]), but is still in its infancy for random Schrödinger operators.

Suppose we have a random Schrödinger operator H which we restrict to a cube Λ_L by appropriate boundary conditions. We call the resulting operator H_{Λ_L}. For a fixed cube Λ_L, the operator H_{Λ_L} has roughly $|\Lambda_L|$ energy levels around an energy E. So we may say that the averaged level spacing near E is $\frac{1}{|\Lambda_L|}$. We now look at the eigenvalues around E under a microscope

zooming the averaged level spacing to 1. The keyword is "unfolding of the spectrum." By this we mean we look at the measure

$$\mu_L([a,b]) = \#\left\{ n; E_n(H_{\Lambda_L}) \in \left[E + \frac{a}{|\Lambda_L|}, E + \frac{b}{|\Lambda_L|} \right] \right\}. \tag{4.21}$$

It is reasonable to ask whether there is a limit of μ_L when L goes to infinity. Moreover, if such a limit exists, what are the properties of the limit measure μ?

Molchanov [**132**] proved the existence of this limit for a one-dimensional model in the continuum. Minami [**131**] investigated the multidimensional discrete case in the regime of Anderson localization. Observe that we have localization for all energies for Molchanov's model.

We discuss Minami's case (Molchanov's case being similar). In the following we give merely a rough sketch of Minami's result, leaving out many details—even assumptions and precise statements. A complete discussion is far beyond the scope of this paper. We urge the reader to look at the paper [**131**] to get a complete picture.

Let us suppose we have an integrated density of states N which has a bounded density n, i.e., $N(E) = \int_{-\infty}^{E} n(\lambda)\, d\lambda$. For energies near the bottom of the spectrum, Minami shows that μ_L is asymptotically a Poisson measure of intensity $n(E)$. This implies that for $a < b$,

$$\mathbb{P}\big(\mu_L([a,b] = k)\big) \approx \frac{n(E)^k (b-a)^k}{k!}\, e^{-n(E)(b-a)}. \tag{4.22}$$

Moreover, the random variables $\mu_L(A_1), \ldots, \mu_L(A_m)$ are approximately independent.

In an informal (but provable) sense, this means that the eigenvalues near E look like independent random variables with a uniform distribution. Especially we have

$$\mathbb{P}\big(\mu_L([a,b] = 1)\big) \approx n(E)(b-a)\, e^{-n(E)(b-a)}, \tag{4.23}$$

$$\mathbb{P}\big(\mu_L([a,b] = 2)\big) \approx \frac{n(E)^2 (b-a)^2}{2}\, e^{-n(E)(b-a)} \tag{4.24}$$

and consequently one may prove

$$\mathbb{P}\bigg(\mu_L([-x/2, x/2]) \le 2 \;\bigg|\; \mu_L([-x/2, x/2]) \le 1 \bigg) \approx \frac{(n(E)^2 x^2)/2}{n(E)\, x} \sim n(E)\, x.$$

This says in a rough way that the *differences* of energy levels near E have a probability density which is strictly positive near 0. In physics terminology, there is no level repulsion.

One expects that this is not the case in the energy region of extended states. So far nobody has proven the existence of extended states for the Anderson model. A fortiori, nobody has proven level repulsion in this case. However, level repulsion is well established for the classical Wigner–Dyson ensembles of random matrix theory ([**124, 29**]).

References

[1] Aizenman, M.; Simon, B.: *Brownian motion and Harnack inequality for Schrödinger operators*. Commun. Pure Appl. Math. **35**, 209–273 (1982).

[2] Akcoglu, M.A.; Krengel, U.: *Ergodic theorems for superadditive processes*. J. Reine Angew. Math. **323**, 53–67 (1981).

[3] Avron, J.; Simon, B.: *Almost periodic Schrödinger operators. II: The integrated density of states*. Duke Math. J. **50**, 369–391 (1983).

[4] Barbaroux, J.M.; Combes, J.M.; Hislop, P.D.: *Localization near band edges for random Schrödinger operators*. Helv. Phys. Acta **70**, 16–43 (1997).

[5] Barlow, M.T.: *Random walks on supercritical percolation clusters*. Ann. Probab. **32**, 3024–3084 (2004).

[6] Benderskij, M.M.; Pastur, L.A.: *On the spectrum of the one-dimensional Schrödinger equation with a random potential*. Math. USSR-Sb. **11**, 245–256 (1970).

[7] Biskup, M.; König, W.: *Long-time tails in the parabolic Anderson model*. Ann. Probab. **29**, 636–682 (2001).

[8] Bingham, N.H.; Goldie, C.M.; Teugels, J.L.: *Regular variation*. Encyclopedia of Mathematics and Its Applications, Vol. 27. Cambridge: Cambridge University Press (1987).

[9] Birman, M.Sh.; Yafaev, D.R.: *The spectral shift function. The work of M. G. Krejn and its further development*. St. Petersbg. Math. J. **4**, 833–870 (1993).

[10] Broderix, K.; Hundertmark, D.; Kirsch, W.; Leschke, H.: *The fate of Lifshits tails in magnetic fields*. J. Stat. Phys. **80**, 1–22 (1995).

[11] de Bruijn, N.G.: *Asymptotic Methods in Analysis*. 2nd edition. Groningen: Wolters–Noordhoff Publishing (1967).

[12] Campanino, M.; Klein, A.: *A supersymmetric transfer matrix and differentiability of the density of states in the one-dimensional Anderson model*. Commun. Math. Phys. **104**, 227–241 (1986).

[13] Campanino, M.; Klein, A.: *Anomalies in the one-dimensional Anderson model at weak disorder*. Commun. Math. Phys. **130**, 441–456 (1990).

[14] Carmona, R.; Klein, A.; Martinelli, F.: *Anderson localization for Bernoulli and other singular potentials*. Commun. Math. Phys. **108**, 41–66 (1987).

[15] Carmona, R.; Lacroix, J.: *Spectral theory of random Schrödinger operators*. Probability and Its Applications. Basel: Birkhäuser Verlag (1990).

[16] Chahrour, A.: *Densité intégrée d'états surfaciques et fonction généralisée de déplacement spectral pour un opérateur de Schrödinger surfacique ergodique*. (Integrated density of surface states and generalized function of spectral shift for an ergodic surface Schrödinger operator). Helv. Phys. Acta **72**, 93–122 (1999).

[17] Chahrour, A.: *On the spectrum of the Schrödinger operator with periodic surface potential*. Lett. Math. Phys. **52**, 197–209 (2000).

[18] Combes, J.M.; Hislop, P.D.: *Localization for some continuous, random Hamiltonians in d-dimensions*. J. Funct. Anal. **124**, 149–180 (1994).

[19] Combes, J.M.; Hislop, P.D.; Klopp, F.: *Hölder continuity of the integrated density of states for some random operators at all energies*. Int. Math. Res. Not. 2003, No. 4, 179–209 (2003).

[20] Combes, J.M.; Hislop, P.D.; Klopp, F.: *An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators*. Preprint.

[21] Combes, J.M.; Hislop, P.D.; Mourre, E.: *Spectral averaging, perturbation of singular spectra, and localization*. Trans. Amer. Math. Soc. **348**, 4883–4894 (1996).

[22] Combes, J.M.; Hislop, P.D.; Nakamura, S.: *The L^p-theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random operators*. Commun. Math. Phys. **218**, 113–130 (2001).

[23] Combes, J.M.; Hislop, P.D.; Klopp, F.; Nakamura, S.: *The Wegner estimate and the integrated density of states for some random operators*. Proc. Indian Acad. Sci. (Math. Sci.) **112**, 31–53 (2002).

[24] Combes, J.M.; Hislop, P.D.; Klopp, F.; Raikov, G.: *Global continuity of the integrated density of states for random Landau Hamiltonians*. Commun. Partial Differ. Equations **29**, 1187–1213 (2004).

[25] Craig, W.; Simon, B.: *Log Hölder continuity of the integrated density of states for stochastic Jacobi matrices*. Commun. Math. Phys. **90**, 207–218 (1983).

[26] Craig, W.; Simon, B.: *Subharmonicity of the Lyapunov index*. Duke Math. J. **50**, 551–560 (1983).

[27] Cycon, H.L.; Froese, R.G.; Kirsch, W.; Simon, B.: *Schrödinger Operators, with Application to Quantum Mechanics and Global Geometry*. (Springer Study ed.) Texts and Monographs in Physics. Berlin: Springer-Verlag (1987).

[28] Damanik, D.; Sims, R.; Stolz, G.: *Localization for one-dimensional, continuum, Bernoulli–Anderson models*. Duke Math. J. **114**, 59–100 (2002).

[29] Deift, P.: *Orthogonal polynomials and random matrices: a Riemann–Hilbert approach*. Courant Lecture Notes in Mathematics, 3. Providence, RI: American Mathematical Society (1999)

[30] Delyon, F.; Souillard, B.: *Remark on the continuity of the density of states of ergodic finite difference operators*. Commun. Math. Phys. **94**, 289–291 (1984).

[31] Dembo, A.; Zeitouni, O.: *Large deviations and applications*. Handbook of Stochastic Analysis and Applications (edited by D. Kannan et al.). New York: Marcel Dekker. Stat., Textb. Monogr. 163, pp. 361–416 (2001).

[32] Deuschel, J.-D.; Stroock, D.: *Large Deviations*. Rev. ed. Pure and Applied Mathematics, 137. Boston: Academic Press (1989).

[33] Donsker, M.; Varadhan S.: *Asymptotic for the Wiener sausage*. Commun. Pure Appl. Math. **28**, 525–565 (1975).

[34] Droese, J.; Kirsch, W.: *The effect of boundary conditions on the density of states for random Schrödinger operators*. Stochastic Processes Appl. **23**, 169–175 (1986).

[35] Eastham, M.S.P.: *The Spectral Theory of Periodic Differential Equations*. Texts in Mathematics. Edinburgh–London: Scottish Academic Press (1973).

[36] Englisch, H.; Kirsch, W.; Schröder, M.; Simon, B.: *The density of surface states*. Phys. Rev. Letters **61**, 1261–1262 (1988).

[37] Englisch, H.; Kirsch, W.; Schröder, M.; Simon, B.: *Random Hamiltonians ergodic in all but one direction*. Commun. Math. Phys. **128**, 613–625 (1990).

[38] Erdös, L.: *Lifschitz tail in a magnetic field: The nonclassical regime*. Probab. Theory Relat. Fields **112**, 321–371 (1998).

[39] Erdös, L.: *Lifschitz tail in a magnetic field: Coexistence of classical and quantum behavior in the borderline case*. Probab. Theory Relat. Fields **121**, 219–236 (2001).

[40] Feller, W.: *An Introduction to Probability Theory and Its Applications. Vol II*. 2nd ed. Wiley Series in Probability and Mathematical Statistics. New York: John Wiley and Sons (1971).

[41] Fischer, W.; Hupfer, T.; Leschke, H.; Müller, P.: *Existence of the density of states for multi-dimensional continuum Schrödinger operators with Gaussian random potentials*. Commun. Math. Phys. **190**, 133–141 (1997)

[42] Fröhlich, J.; Martinelli, F.; Scoppola, E.; Spencer, T.: *Constructive proof of localization in the Anderson tight binding model*. Commun. Math. Phys. **101**, 21–46 (1985).

[43] Fröhlich, J.; Spencer, T.: *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*. Commun. Math. Phys. **88**, 151–184 (1983).

[44] Fukushima, M.: *On asymptotics of spectra of Schrödinger equations*. Aspects Statistiques et Aspects Physiques des Processus Gaussiens (Saint-Flour, 1980). Colloq. Int. C.N.R.S., 307, pp. 335–347 (1981).

[45] Fukushima, M.; Nagai, H.; Nakao, S.: *On an asymptotic property of spectra of a random difference operator*. Proc. Japan Acad. **51**, 100–102 (1975).
[46] Fürstenberg, H.; Kesten, H.: *Products of random matrices*, Ann. Math. Stat. **31**, 457–469 (1960).
[47] Gärtner, J.; König, W.: *Moment asymptotics for the continuous parabolic Anderson model*. Ann. Appl. Probab. **10**, 192–217 (2000).
[48] Gärtner, J.; Molchanov, S.: *Parabolic problems for the Anderson model. I: Intermittency and related topics*. Commun. Math. Phys. **132**, 613–655 (1990).
[49] Gärtner, J.; Molchanov, S.: *Parabolic problems for the Anderson model. II: Second-order asymptotics and structure of high peaks*. Probab. Theory Relat. Fields **111**, 17–55 (1998).
[50] Gärtner, J.; Molchanov, S.: *Moment asymptotics and Lifshitz tails for the parabolic Anderson model*. Stochastic Models (Ottawa, 1998). Edited by Luis G. Gorostiza et al. Providence, RI: American Mathematical Society for the Canadian Mathematical Society. CMS Conf. Proc. 26, pp. 141–157 (2000).
[51] Goldstein, M.; Schlag, W.: *Hölder continuity of the integrated density of states for quasiperiodic Schrödinger equations and averages of shifts of subharmonic functions*. Ann. of Math. (2) **154**, 155–203 (2001).
[52] Goldstein, M.; Schlag, W.: *On the formation of gaps in the spectrum of Schroedinger operators with quasi-periodic potentials*. In this Festschrift.
[53] Goldstein, M.; Schlag, W.: *Fine properties of the integrated density of states and a quantitative separation property of the Dirichlet eigenvalues*. Preprint 2005.
[54] Grimmett, G.: *Percolation*, 2nd ed. New York: Springer (1999).
[55] Halperin, B.: *Properties of a particle in a one-dimensional random potential*. Adv. Chem. Phys. **31**, 123–177 (1967).
[56] Herbert, D.; Jones, R.: *Localized states in disordered systems*. J. Phys. C **4**, 1145–1161 (1971).
[57] Hislop, P.D.; Klopp, F.: *The integrated density of states for some random operators with nonsign definite potentials*. J. Funct. Anal. **195**, 12–47 (2002).
[58] den Hollander, F.: *Large Deviations*. Fields Institute Monographs, 14. Providence, RI: American Mathematical Society (2000).
[59] Hundertmark, D.; Killip, R.; Nakamura, S.; Stollmann, P.; Veselić, I.: *Bounds on the spectral shift function and the density of states*. Comm. Math. Phys. **262**, 489–503 (2005).
[60] Hundertmark, D.; Kirsch, W.; Warzel, S.: *Classical magnetic Lifshits tails in three space dimensions: Impurity potentials with slow anisotropic decay*. Markov Process. Relat. Fields **9**, 651–660 (2003).
[61] Hundertmark, D.; Simon, B.: *An optimal L^p-bound on the Krein spectral shift function*. J. Anal. Math. **87**, 199–208 (2002).
[62] Hupfer, T.; Leschke, H.; Müller, P.; Warzel, S.: *Existence and uniqueness of the integrated density of states for Schrödinger operators with magnetic fields and unbounded random potentials*. Rev. Math. Phys. **13**, 1547–1581 (2001).
[63] Hupfer, T.; Leschke, H.; Müller, P.; Warzel, S.: *The absolute continuity of the integrated density of states for magnetic Schrödinger operators with certain unbounded random potentials*. Commun. Math. Phys. **221**, 229–254 (2001).
[64] Hupfer, T.; Leschke, H.; Warzel, S.: *Poissonian obstacles with Gaussian walls discriminate between classical and quantum Lifshits tailing in magnetic fields*. J. Stat. Phys. **97**, 725–750 (1999).
[65] Javrjan, V.A.: *A certain inverse problem for Sturm-Liouville operators*. (Russian) Izv. Akad. Nauk Ar-mjan. SSR Ser. Mat. **6**, 246–251 (1971).
[66] Kingman, J: *Subadditive ergodic theory*. Ann. Probab. **1**, 883–909 (1973).
[67] Kirsch, W.: *Small perturbations and the eigenvalues of the Laplacian on large bounded domains*. Proc. Amer. Math. Soc. **101**, 509–512 (1987).

[68] Kirsch, W.: *Random Schrödinger operators*. Schrödinger Operators (Sønderborg, Denmark, 1988). Lect. Notes in Phys., 345, pp. 264–370 (1989).
[69] Kirsch, W.: *Wegner estimates and Anderson localization for alloy-type potentials*. Math. Z. **221**, 507–512 (1996).
[70] Kirsch, W.: *An invitation to random Schrödinger operators*. Preprint (2006).
[71] Kirsch, W.; Khorunzhiy, O.; Müller, P.: *Lifshits tails for spectra of Erdös–Rényi random graphs*. Ann. Appl. Probab. **16**, 295–309 (2006).
[72] Kirsch, W.; Klopp, F.: *The band-edge behavior of the density of surfacic states*. Math. Phys. Anal. Geom. To appear.
[73] Kirsch, W.; Leschke, H.; Warzel, S.: *On Lifshitz tails for homogeneous magnetic fields in three dimensions*. Unpublished.
[74] Kirsch, W.; Martinelli, F.: *On the density of states of Schrödinger operators with a random potential*. J. Phys. A **15**, 2139–2156 (1982).
[75] Kirsch, W.; Martinelli, F.: *On the spectrum of Schrödinger operators with a random potential*. Commun. Math. Phys. **85**, 329–350 (1982).
[76] Kirsch, W.; Martinelli, F.: *On the ergodic properties of the spectrum of general random operators*. J. Reine Angew. Math. **334**, 141–156 (1982).
[77] Kirsch, W.; Martinelli, F.: *Large deviations and Lifshitz singularity of the integrated density of states of random Hamiltonians*. Commun. Math. Phys. **89**, 27–40 (1983).
[78] Kirsch, W.; Müller, P.: *Spectral properties of the Laplacian on bond-percolation graphs*. Math. Z. **252**, 899–916 (2006).
[79] Kirsch, W.; Nitzschner, F.: *Lifshitz-tails and non-Lifshitz-tails for one-dimensional random point interactions*, Oper. Theory, Adv. Appl. **46**, 171–178 (1990).
[80] Kirsch, W.; Simon, B.: *Lifshitz tails for periodic plus random potentials*. J. Stat. Phys. **42**, 799–808 (1986).
[81] Kirsch, W.; Simon, B.; *Comparison theorems for the gap of Schrödinger operators*. J. Funct. Anal. **75**, 396–410 (1987).
[82] Kirsch, W.; Stollmann, P.; Stolz, G.: *Anderson localization for random Schrödinger operators with long range interactions*. Commun. Math. Phys. **195**, 495–507 (1998).
[83] Kirsch, W.; Stollmann, P.; Stolz, G.: *Localization for random perturbations of periodic Schrödinger operators*. Random Oper. Stoch. Equ. **6**, 241–268 (1998).
[84] Kirsch, W.; Veselic, I.: *Existence of the density of states for one-dimensional alloy-type potentials with small support*. Mathematical Results in Quantum Mechanics (Taxco, Mexico, 2001). Rdited by Ricardo Weder et al. Contemp. Math. **307**, 171–176 (2002). Providence, RI: American Mathematical Society.
[85] Kirsch, W.; Veselic, I.: *Wegner estimate for sparse and other generalized alloy type potentials*. Proc. Indian Acad. Sci. (Math. Sci.) **112**, 131–146 (2002).
[86] Kirsch, W.; Warzel, S.: *Lifshits tails caused by anisotropic decay: the emergence of a quantum-classical regime*. Math. Phys. Anal. Geom. **8**, 257–285 (2005).
[87] Kirsch, W.; Warzel, S.: *Anderson localization and Lifshits tails for random surface potentials*. J. Funct. Anal. **230**, 222–250 (2006).
[88] Klassert, S.: *Unschärfe, der supersymmetrische Replica Trick und Lokalisierung*, Diploma Thesis, Johann Wolfgang Goethe-Universität, Frankfurt (2001).
[89] Kleespies, F.; Stollmann, P.: *Lifshitz asymptotics and localization for random quantum waveguides*. Rev. Math. Phys. **12**, 1345–1365 (2000).
[90] Klein, A.: *The supersymmetric replica trick and smoothness of the density of states for random Schrödinger operators*. Operator Theory, Operator Algebras and Applications, (Durham, NH, 1988). Proc. Symp. Pure Math., 51, Part 1, pp. 315-331 (1990).
[91] Klein, A.; Lacroix, J.; Speis, A.: *Regularity of the density of states in the Anderson model on a strip for potentials with singular continuous distributions*. J. Stat. Phys. **57**, 65–88 (1989).

[92] Klein, A.; Martinelli, F.; Perez, J.F.: *A rigorous replica trick approach to Anderson localization in one dimension*. Commun. Math. Phys. **106**, 623–633 (1986).
[93] Klein, A.; Speis, A.: *Smoothness of the density of states in the Anderson model on a one-dimensional strip*. Ann. Phys. **183**, 352–398 (1988).
[94] Klein, A.; Speis, A.: *Regularity of the invariant measure and of the density of states in the one-dimensional Anderson model*. J. Funct. Anal. **88**, 211–227 (1990).
[95] Klopp, F.: *Localization for some continuous random Schrödinger operators*. Commun. Math. Phys. **167**, 553–569 (1995).
[96] Klopp, F.: *Band edge behavior of the integrated density of states of random Jacobi matrices in dimension 1*. J. Stat. Phys. **90**, 927–947 (1998).
[97] Klopp, F.: *Internal Lifshits tails for random perturbations of periodic Schrödinger operators*. Duke Math. J. **98**, 335–396 (1999).
[98] Klopp, F.: *Precise high energy asymptotics for the integrated density of states of an unbounded random Jacobi matrix*. Rev. Math. Phys. **12**, 575–620 (2000).
[99] Klopp, F.: *Weak disorder localization and Lifshitz tails: Continuous Hamiltonians*. Ann. Henri Poincaré **3**, 711–737 (2002).
[100] Klopp, F.: *Lifshitz tails for random perturbations of periodic Schrödinger operators*. Proc. Indian Acad. Sci. (Math. Sci.) **112**, 147–162 (2002).
[101] Klopp, F.; Pastur, L.: *Lifshitz tails for random Schrödinger operators with negative singular Poisson potential*. Commun. Math. Phys. **206**, 57–103 (1999).
[102] Klopp, F.; Raikov, G.: *Lifshitz tails in constant magnetic fields*. To appear in Commun. Math. Phys.
[103] Klopp, F.; Ralston, J.: *Endpoints of the spectrum of periodic operators are generically simple*. Methods Appl. Anal. **7**, 459–463 (2000).
[104] Klopp, F.; Wolff, T.: *Internal Lifshitz tails for random Schrödinger operators*. XIIIth International Congress on Mathematical Physics (Imperial College, London, 2000). pp. 425-433. Edited by A. Grigoryan et al. Boston: International Press (2001).
[105] Klopp, F.; Wolff, T.: *Lifshitz tails for 2-dimensional random Schrödinger operators*. J. Anal. Math. **88**, 63–147 (2002).
[106] Kostrykin, V.; Schrader, R.: *The density of states and the spectral shift density of random Schrödinger operators*. Rev. Math. Phys. **12**, 807–847 (2000).
[107] Kostrykin, V.; Schrader, R.: *Regularity of the surface density of states*. J. Funct. Anal. **187**, 227–246 (2001).
[108] Kostrykin, V.; Veselic, I.: *On the Lipschitz continuity of the integrated density of states for sign-indefinite potentials*. Math. Z. **252**, 367–392 (2006).
[109] Kotani, S.: *Lyapunov exponents and spectra for one-dimensional random Schrödinger operators*. Random Matrices and Their Applications (Brunswick, Maine, 1984). Contemp. Math. **50**, 277–286 (1986).
[110] Kotani, S.; Simon, B.: *Localization in general one-dimensional random systems. II. Continuum Schrödinger operators*. Commun. Math. Phys. **112**, 103–119 (1987).
[111] Krengel, U.: *Ergodic Theorems*. De Gruyter Studies in Mathematics, 6. Berlin–New York: Walter de Gruyter (1985).
[112] Kuchment, P.: *Floquet Theory for Partial Differential Equations*. Operator Theory: Advances and Applications, 60. Basel: Birkhäuser Verlag (1993).
[113] Lamperti, J.: *Stochastic Processes. A Survey of the Mathematical Theory*. Applied Mathematical Sciences, Vol. 23. New York–Heidelberg: Springer (1977).
[114] LePage, E.: *Répartition d'état d'un opérateur de Schrödinger aléatoire. Distribution empirique des valeurs propres d'une matrice de Jacobi*. (State distribution of a random Schrödinger operator. Empirical distribution of the eigenvalues of a Jacobi matrix) Probability Measures on Groups, VII (Oberwolfach, 1983). Lecture Notes in Math., 1064, pp. 309–367. Berlin: Springer (1984).

[115] Leschke, H.; Müller, P.; Warzel, S. *A survey of rigorous results on random Schrödinger operators for amorphous solids*. Interacting Stochastic Systems. Edited by Jean-Dominique Deuschel et al. pp. 119–151. Berlin: Springer (2005).
[116] Leschke, H.; Warzel, S.: *Quantum-classical transitions in Lifshits tails with magnetic fields*. Phys. Rev. Lett. **92**, 086402 (2004).
[117] Lifshitz, I.: *Structure of the energy spectrum of impurity bands in disordered solid solutions*. Soviet Physics JETP **17**, 1159–1170 (1963).
[118] Lifshitz, I.: *Energy spectrum structure and quantum states of disordered condensed systems*. Soviet Physics Usp. **7**, 549–573 (1965).
[119] Lifshitz, I.; Gredeskul, S.A.; Pastur, L.A.: *Introduction to the Theory of Disordered Systems*. New York: Wiley (1988).
[120] March, P.; Sznitman, A.-S.: *Some connections between excursion theory and the discrete Schrödinger equation with random potentials*. Probab. Theory Relat. Fields **75**, 111–53 (1987).
[121] Martinelli, F.; Holden, H.: *On absence of diffusion near the bottom of the spectrum for a random Schrödinger operator on* $L^2(\mathbb{R}^\nu)$. Commun. Math. Phys. **93**, 197–217 (1984).
[122] Martinelli, F.; Micheli L.: *On the large coupling constant behavior of the Lyapunov exponent of a binary alloy*. J. Stat. Phys. **48**, 1–18 (1986).
[123] McKean, H.P.; Trubowitz, E.: *Hill's surfaces and their theta functions*. Bull. Amer. Math. Soc. **84**, 1042–1085 (1978).
[124] Mehta, M.L.: *Random Matrices*. Boston: Academic Press (1991).
[125] Metzger, B.: *Asymptotische Eigenschaften im Wechselspiel von Diffusion und Wellenausbreitung in zufälligen Medien*, Ph.D. thesis, TU Chemnitz (2005).
[126] Metzger, B.: *Lifshitz asymptotics for the Anderson model: On the transition from classical to quantum regime*. In preparation.
[127] Metzger, B.: *Tauberian estimates from below and above*. In preparation.
[128] Mezincescu, G.A.: *Internal Lifschitz singularities of disordered finite-difference Schrödinger operators*. Commun. Math. Phys. **103**, 167–176 (1986).
[129] Mezincescu, G.A.: *Lifschitz singularities for periodic operators plus random potentials*. J. Stat. Phys. **49**, 1181–1190 (1987).
[130] Mezincescu, G.A.: *Internal Lifschitz singularities for one-dimensional Schrödinger operators*. Commun. Math. Phys. **158**, 315–325 (1993).
[131] Minami, N.: *Local fluctuation of the spectrum of a multidimensional Anderson tight binding model*. Commun. Math. Phys. **177**, 709–725 (1996).
[132] Molchanov, S.A.: *The local structure of the spectrum of the one-dimensional Schrödinger operator*. Commun. Math. Phys. **78**, 429–446 (1981).
[133] Müller, P.; Stollmann, P.: *Spectral asymptotics of the Laplacian on supercritical bond-percolation graphs*. Preprint (2005).
[134] Najar, H.: *Asymptotique de la densité d'états intégrée des modéles aléatoires continus*. Thèse Université Paris 13 (2000).
[135] Najar, H.: *Lifshitz tails for random acoustic operators*. J. Math. Phys. **44**, 1842–1867 (2003).
[136] Najar, H.: *Asymptotic behavior of the integrated density of states of acoustic operators with random long range perturbations*. J. Stat. Phys. **115**, 977–996 (2004).
[137] Nakamura, S.: *Spectral shift function for trapping energies in the semiclassical limit*. Commun. Math. Phys. **208**, 173–193 (1999).
[138] Nakamura, S.: *Lifshitz tail for 2D discrete Schrödinger operator with random magnetic field*. Ann. Henri Poincaré **1**, 823–835 (2000).
[139] Nakamura, S.: *Lifshitz tail for Schrödinger operator with random magnetic field*. Commun. Math. Phys. **214**, 565–572 (2000).
[140] Nakamura, S.: *A remark on the Dirichlet–Neumann decoupling and the integrated density of states*. J. Funct. Anal. **179**, 136–152 (2001).

[141] Nakamura, S.: *A remark on the Lifshitz tail for Schrödinger operator with random magnetic field*. Proc. Indian Acad. Sci. (Math. Sci.) **112**, 183–187 (2002).
[142] Nakao, S.: *On the spectral distribution of the Schrödinger operator with random potential*. Jap. J. Math. (N.S.) **3**, 111–139 (1977).
[143] Pastur, L.A.: *Selfaverageability of the number of states of the Schrödinger equation with a random potential*. (Russian) Mat. Fiz. i Funkcional. Anal. Vyp. **2**, 111–116, 238 (1971).
[144] Pastur, L.A.: *On the distribution of the eigenvalues of the Schrödinger equation with a random potential*. Funct. Anal. Appl. **6**, 163–165 (1972).
[145] Pastur, L.A.: *Spectra of random selfadjoint operators*. Russian Math. Surv. **28**, 1–67 (1973).
[146] Pastur, L.A.: *Behavior of some Wiener integrals as $t \to \infty$ and the density of states of Schrödinger equations with random potential*. Theor. Math. Phys. **32**, 615–620 (1978).
[147] Pastur, L.A.: *Spectral properties of disordered systems in the one-body approximation*. Commun. Math. Phys. **75**, 179–196 (1980).
[148] Pastur, L.; Figotin, A.: *Spectra of Random and Almost-Periodic Operators*. Grundlehren der Mathematischen Wissenschaften, 297. Berlin: Springer-Verlag (1992).
[149] Reed, M.; Simon, B.: *Methods of Modern Mathematical Physics. II: Fourier Analysis, Self-Adjointness*. New York: Academic Press (1975).
[150] Reed M.; Simon B.: *Methods of Modern Mathematical Physics. IV: Analysis of Operators*. New York: Academic Press (1978).
[151] Reed, M.; Simon, B.: *Methods of Modern Mathematical Physics. III: Scattering Theory*. New York: Academic Press (1979).
[152] Reed, M.; Simon, B.: *Methods of Modern Mathematical Physics. I: Functional Analysis*. Revised and enlarged ed. New York: Academic Press (1980).
[153] Shubin, C.; Vakilian, R.; Wolff, T.: *Some harmonic analysis questions suggested by Anderson–Bernoulli models*. Geom. Funct. Anal. **8**, 932–964 (1998).
[154] Simon, B.: *Functional Integration and Quantum Physics*. Pure and Applied Mathematics, 86. New York: Academic Press (1979); 2nd ed. Providence, RI: AMS Chelsea Publishing (2005).
[155] Simon, B.: *Schrödinger semigroups*. Bull. Amer. Math. Soc. (N.S.) **7**, 447–526 (1982).
[156] Simon, B.: *Lifshitz tails for the Anderson model*. J. Phys. **38**, 65–76 (1985).
[157] Simon, B.: *Internal Lifschitz tails*. J. Stat. Phys. **46**, 911–918 (1987).
[158] Simon, B.: *Spectral analysis of rank one perturbations and applications*. Mathematical Quantum Theory II: Schrödinger Operators (Vancouver, 1993). Edited by J. Feldman et al. Providence, RI: American Mathematical Society. CRM Proc. Lect. Notes. **8**, 109–149 (1995).
[159] Simon, B.: *Spectral averaging and the Krein spectral shift*. Proc. Amer. Math. Soc. **126**, 1409–1413 (1998).
[160] Simon, B.: *Schrödinger operators in the twenty-first century*, XIIIth International Congress on Mathematical Physics (Imperial College, London, 2000). Edited by A. Fokas et al. pp. 283–288. Boston: International Press (2001).
[161] Simon, B.; Taylor, M.: *Harmonic analysis on $SL(2, \mathbb{R})$ and smoothness of the density of states in the one-dimensional Anderson model*, Commun. Math. Phys. **101**, 1–19 (1985).
[162] Simon, B.; Taylor, M.; Wolff, T.: *Some rigorous results for the Anderson model*, Phys. Rev. Lett. **54**, 1589–1592 (1985).
[163] Simon, B.; Wolff, T.: *Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians*. Commun. Pure Appl. Math. **39**, 75–90 (1986).

[164] Stollmann, P.: *Lifshitz asymptotics via linear coupling of disorder*. Math. Phys. Anal. Geom. **2**, 279–289 (1999).
[165] Stollmann, P.: *Wegner estimates and localization for continuum Anderson models with some singular distributions*. Arch. Math. **75**, 307–311 (2000).
[166] Stollmann, P.: *Caught by disorder*. Bound States in Random Media. Progress in Mathematical Physics, 20. Boston: Birkhäuser (2001).
[167] Sznitman, A.-S.: *Lifschitz tail and Wiener sausage on hyperbolic space*. Commun. Pure Appl. Math. **42**, 1033–1065 (1989).
[168] Sznitman, A.-S.: *Lifschitz tail and Wiener sausage*. J. Funct. Anal. **94**, 223–246, 247–272 (1990).
[169] Sznitman, A.-S.: *Lifschitz tail on hyperbolic space: Neumann conditions*. Commun. Pure Appl. Math. **43**, 1–30 (1990).
[170] Sznitman, A.-S.: *Brownian Motion, Obstacles and Random Media*. Springer Monographs in Mathematics. Berlin: Springer (1998).
[171] Thouless, D.J.: *Electrons in disordered systems and the theory of localization*. Phys. Rep. **13**, 93–142 (1974).
[172] Ueki, N.: *On spectra of random Schrödinger operators with magnetic fields*. Osaka J. Math. **31**, 177–187 (1994).
[173] Ueki, N.: *Simple examples of Lifschitz tails in Gaussian random magnetic fields*. Ann. Henri Poincaré **1**, 473–498 (2000).
[174] Ueki, N.: *Lifschitz tail of a random Pauli Hamilitonian with an anomalous moment*. Jap. J. Math. (N.S.) **28**, 261–286 (2002).
[175] Veselic, I.: *Localization for random perturbations of periodic Schrödinger operators with regular Floquet eigenvalues*. Ann. Henri Poincaré **3**, 389–409 (2002).
[176] Veselic, I.: *Spectral analysis of percolation Hamiltonians*. Math. Ann. **331**, 841–865 (2005).
[177] Veselic, I.: *Integrated density and Wegner estimates for random Schrödinger operators*. Spectral Theory of Schrödinger Operators (Mexico, 2001). Edited by Rafael del Río et al. Providence, RI: American Mathematical Society (AMS). Contemp. Math. **340**, 97–183 (2004).
[178] Warzel, S.: *On Lifshitz tails in magnetic fields*. Doctoral thesis, Universität Erlangen–Nürnberg, Logos, Berlin (2001).
[179] Wegner, F.: *Bounds on the density of states in disordered systems*. Z. Phys. B **44**, 9–15 (1981).
[180] Yafaev, D.: *Mathematical scattering theory. General theory*. Translations of Mathematical Monographs, 105. Providence, RI: American Mathematical Society (1992).

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